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# A NOTE ON THE STABILITY OF EQUILIBRIUM STATE OF NON-CONSERVATIVE MECHANICAL SYSTEMS

*A b s t r a c t*

The note investigates the stability of equilibrium state of a nonlinear mechanical system subjected to potential, dissipative, gyroscopic and circulatory forces. Two stability theorems are stated and proved. These theorems supplement previously published results in this field [2-5], [9].

# NAPOMENA O STABILNOSTI RAVNOTEŽNOG STANJA NEKONZERVATIVNIH MEHANIČKIH SISTEMA

*I z v o d*

Istražuje se stabilnost ravnotežnog stanja nelinearnog mehaničkog sistema podvrgnutog dejstvu potencijalnih, disipativnih, giroskopskih i cirkulacionih sila. Formulisana su i dokazana su dva stava koja dopunjavaju ranije publikovane rezultate iz ove oblasti [2-5], [9].

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## INTRODUCTION

The dynamical behaviour of a mechanical system with dissipative, gyroscopic, circulatory and potential forces can be described in the vicinity of the equilibrium by a vector differential equation of the form

$$\ddot{x} + (D+G)\dot{x} + (N+K)x = X(x,\dot{x}) \tag{1}$$

Here, $x$ is the $n$-dimensional position vector, the damping matrix $D$ is real positive semi-definite, the potential force matrix $K$ is real symmetric, and the gyroscopic matrix $G$ and the circulatory matrix $N$ are real skew-symmetric, and $X(x,\dot{x})$ is a collection of terms of no lower than second order in $x, \dot{x}$.

The stability of the equilibrium state

$$x = 0, \dot{x} = 0 \tag{2}$$

of system (1) has received increasing attention recently because of interest in robotic and large space structures. The standard tool for stability investigations is to apply the Routh-Hurwitz criterion (see [1]) to the system (1) involving the characteristic polynomial of degree $2n$. But it is somewhat cumbersome when the degree of freedom is large, and alternative criteria such as those which provide simpler conditions directly in terms of the coefficient matrices prove to be more attractive. Such criteria are not well developed, and these are as follows:

(A) If $4K - G^2 < 0$ , then the equilibrium state (2) is unstable [2] (see also [3]);

(B) If $D > 0$, and the matrices $ND^{-1}$, $ND^{-1}G$ and $ND^{-1}K$ are symmetric and

$$(ND^{-1})^2 - ND^{-1}G + K > 0$$

then the equilibrium state (2) is asymptotically stable [4].

(C) If $K \geq 0, \det(K+N) \neq 0, D > 0$ and

$$2\lambda_{max}(iN) < \lambda_{min}(D)\left[\sqrt{\lambda_{min}^2(D) + 4\lambda_{min}(K)} - \sqrt{\lambda_{max}^2(D) + \lambda_{max}^2(iG)}\right],$$

where $\lambda(.)$ denotes eigenvalue of the enclosed matrix, the equilibrium state (2) is asymptotically stable [5].

These results are supplemented by two statements in the following section.

## STABILITY THEOREMS

The second-order equation (1) can be rewritten in first-order form

$$\dot{y} = Ay + Y(y) \tag{3}$$

with

$$y = \begin{bmatrix} \dot{x} \\ -- \\ x \end{bmatrix}, A = \begin{bmatrix} -(D+G) & | & -(N+K) \\ ------ & | & ------ \\ I & | & 0 \end{bmatrix}, Y = \begin{bmatrix} X \\ -- \\ 0 \end{bmatrix} \tag{4}$$

The characteristic equation of the linear approximation of (3) (or (1)) is

$$\lambda^{2n} + a_1\lambda^{2n-1} + a_2\lambda^{2n-2} + \ldots + a_{2n-1}\lambda + a_{2n} = 0 \tag{5}$$

in which $a_k(k = 1, \ldots, 2n)$ (k=1,...,2n) can be determined through the following formulae [6]

$$a_k = -\frac{1}{k}Tr(A_k) \tag{6}$$

where

$$A_k = A^k + a_1A^{k-1} + \ldots + a_{k-1}A, k = 1, 2, \ldots \tag{7}$$

and $Tr(A_k)$ denotes the trace of $A_k$.

Denote Euclidean norm of matrix $B$ by $||B||$, i. e.,

$$||B|| = \left(\sum_{i,j} |b_{ij}|^2\right)^{1/2}$$

**Theorem 1.** *If*

$$Tr^2(D) - ||D||^2 + 2Tr(K) + ||G||^2 < 0, \tag{8}$$

*then the equilibrium state (2) of system (1) is unstable, no matter how circulatory forces act on the system.*

When $D = 0$ and $N = 0$ (conservative gyroscopic system), this theorem coincides with recent result [7].

Proof. From (4) and (6), we have $a_1 = -Tr(A) = Tr(D)$ and

$$a_2 = -\frac{1}{2}Tr(A^2+a_1A) = \frac{1}{2}(Tr^2(D)-Tr(D^2))+Tr(K)-\frac{1}{2}Tr(G^2) \tag{9}$$

Since $G^T = -G$ there exists an orthogonal matrix $T$ such that [8]

$$T^T GT = \widetilde{G} = diag\left(g_1 \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \dots, g_k \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, 0, \dots, 0\right), k \leq [n/2] \quad (10)$$

and

$$T^T G^2 T = \widetilde{D}^2 = -diag(g_1^2, g_1^2, \dots, g_k^2, g_k^2, 0 \dots, 0) \quad (11)$$

From (10) and (11), we have

$$Tr(G^2) = -||G||^2 \quad (12)$$

because orthogonal transformation preserve Euclidean norm and trace of a matrix. Similarly

$$Tr(D^2) = ||D||^2 \quad (13)$$

It follows from (9), (12) and (13) that (8) is equivalent to the condition $a_2 < 0$, which implies that at least one of the roots of characteristic equation (5) has a positive real part. Then, under condition (8), according to Liapunov's theorem on the stability in the first approximation [1], the equilibrium state (2) of system (1) is unstable.

Theorem 1 is a new criterion for instability and the following simple examples show that neither condition $(A)$ nor condition (8) implies the other one.

Example 1. Let

$$D = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, G = N = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \text{ and } K = \begin{bmatrix} 2 & -1 \\ -1 & -6 \end{bmatrix}.$$

We have

$$4K - G^2 = \begin{bmatrix} 9 & -4 \\ -4 & -23 \end{bmatrix} \not> 0$$

so that criterion $(A)$ does not apply. However,

$$Tr^2(D) - ||D||^2 + 2Tr(K) + ||G||^2 = -2$$

and instability follows from Theorem 1.

Example 2. Let

$$D = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}, G = N = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \text{ and } K = \begin{bmatrix} -2 & 1 \\ 1 & -1 \end{bmatrix}.$$

We have

$$4K - G^2 = \begin{bmatrix} -7 & 4 \\ 4 & -3 \end{bmatrix} < 0$$

and instability follows from criterion $(A)$. On the other hand,

$$Tr^2(D) - ||D||^2 + 3Tr(K) + ||G||^2 = 4$$

and Theorem 1 tells us nothing.

Our next result concerns systems (1) with an additional positive parameter $s$:

$$\ddot{x} + (D + G)\dot{x} + (sN + K)x = X(x, \dot{x}) \tag{14}$$

It is well known that the equilibrium state (2) of system (14) is unstable if $\det N \neq 0$ (then n is even) and if the positive real number $s$ is sufficiently large [9].

**Theorem 2.** *If* $Tr(NG) \neq 0$, *then for sufficiently large positive real number the equilibrium state (2) of the system (14) is unstable.*

Proof. From relations (4) to (7), we conclude that

$$a_3 = -sTr(NG) + \hat{a} \tag{15}$$

and

$$\Delta_2 = a_1 a_2 - a_3 = sTr(NG) + \tilde{a} \tag{16}$$

in which terms $\hat{a}$ and $\tilde{a}$ do not depend on $s$. If $Tr(NG) > 0$, then $a_3 < 0$ for sufficiently large $s$, and, consequently, at least one of the roots of equation (5) has a positive real part. If $Tr(NG) < 0$, then $\Delta_2 < 0$ for sufficiently large $s$, and, according to Routh-Hurwitz theorem there is at least one root of equation (5) with positive real part.

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