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MULTIPLICATIVE DECOMPOSITION OF DEFORMATION GRADIENT IN CONTINUUM MECHANICS: THERMOELASTICITY, ELASTOPLASTICITY AND BIOMECHANICS

A b s t r a c t

A survey of the application of the multiplicative decomposition of deformation gradient in the constitutive theory of material response is given with a focus on the problems of finite-deformation thermoelasticity, elastoplasticity, and biomechanics. Constitutive equations of isotropic thermoelasticity are derived in general and in the case of quadratic dependence of the strain energy on the finite elastic strain. Polycrystalline and single crystal constitutive formulations of elastoplastic response are reviewed, with a particular accent given to additive decompositions of the stress and strain rates into their elastic and plastic parts. The framework of the multiplicative decomposition of deformation gradient is again utilized. An analysis of the stress-modulated growth of pseudo-elastic soft tissues modeled as isotropic materials is then given. A structure of the evolution equation for the growth-induced stretch ratio is discussed in the context of the rate-type biomechanic theory.

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MULTIPLIKATIVNA DEKOMPOZICIJA DEFORMACIONOG GRADIJENTA U MEHANICI KONTINUUMA: TERMOELASTIČNOST, ELASTOPLASTIČNOST I BIOMEHANIKA

Izvod

Primjena dekompozicije deformacionog gradijenta u konstitutivnoj teoriji ponašanja materijala je analizirana uz poseban naglasak na probleme velikih deformacija u termoelastičnosti, elastoplastičnosti i biomehanici. Konstitutivne jednačine izotropne termoelastičnosti su izvedene u opštem i u slučaju kvadratne zavisnosti deformacione energije of tenzora konačne elastične deformacije. Formulacija polikristalne i monokristalne teorije plastičnosti je data, uz posebnu pažnju posvećenu aditivnoj dekompoziciji izvoda napona i deformacije na bazi multiplikativne dekompozicije elastoplastičnog deformacionog gradijenta. Analiza izotropnog rasta pseudo-elastičnih mekih tkiva pod uticajem napona je zatim prikazana. Evoluciona jednačina za relativno izduženje usljed porasta mase je diskotovana u kontekstu biomehaničke teorije izvodnog tipa.

1. INTRODUCTION

The objective of this article is to give an overview and comparative analysis of the application of the multiplicative decomposition of deformation gradient in constitutive theories of material response. This decomposition is based on the introduction of an intermediate configuration by conceptual destressing of the deformed configuration to zero stress. A significance of such configuration in material modeling was indicated by Eckart (1948), Kröner (1960), and Sedov (1962), but its definite introduction in non-linear continuum mechanics is attributed to Stojanović *et al.* (1964) in the case of finite deformation thermoelasticity, and to Lee (1969) in the case of finite deformation elastoplasticity. The decomposition was also extensively utilized in single crystal plasticity. More recently, following the work of Rodrigez *et al.* (1996), the

intermediate configuration and the multiplicative decomposition of deformation gradient found its use in the constitutive analysis of the stress-modulated growth of soft tissues in biomechanics. A survey of the application of the decomposition in these three areas of continuum mechanics is presented in this paper, mostly based on the author's own involvement and contributions in this field.

The formulation of the constitutive theory of finite thermoelasticity is first described. The intermediate configuration is introduced by a conceptual isothermal destressing of the current configuration to zero stress. The total deformation gradient is then decomposed into the product of purely elastic and thermal parts. Stojanović *et al.* (1964, 1970) used this approach to study both non-polar and polar thermoelastic materials. However, in contrast to the decomposition of elastoplastic deformation gradient, the decomposition of thermoelastic deformation gradient received less attention in the mechanics community, although there has been recently some revived interest shown in the work of Imam and Johnson (1998), and Vujošević and Lubarda (2001). The analysis from the latter contribution is followed in the presentation in Section 2. For simplicity, the considerations are restricted to elastically and thermally isotropic materials, although an extension of the analysis to transversely isotropic and orthotropic materials is indicated. Particular attention is given to quadratic dependence of the elastic strain energy on the finite elastic strain.

The fundamental aspects of the finite deformation elastoplasticity within the framework of the multiplicative decomposition are then presented. The intermediate configuration, again obtained from the deformed configuration by isothermal elastic destressing to zero stress, differs from the initial configuration by a residual (plastic) deformation, and from the current configuration by a reversible (elastic) deformation. The corresponding decomposition of the elastoplastic deformation gradient into its elastic and plastic part was introduced by Lee (1969). Related early contributions include Backman (1964), Lee and Liu (1967), Fox (1968), Willis (1969), Mandel (1971,1973), and Kröner and Teodosiu (1973). The decomposition was extensively used in the

phenomenological theory of plasticity during past three decades (Lubarda, 2002). It was also employed in the single crystal plasticity, assuming that crystallographic slip is the only mechanism of plastic deformation. The plastic part of deformation gradient in this model is due to slip, while elastic part accounts for the lattice stretching and rotation; Asaro and Rice (1977), Hill and Havner (1982), Asaro (1983), Havner (1992). Essential features of the polycrystalline and single crystal elastoplastic constitutive formulations within the framework of the multiplicative decomposition are presented in Section 3. The kinematic and kinetic aspects of the additive decompositions of the stress and strain rates into their elastic and plastic parts are given.

The third area in which the multiplicative decomposition of deformation gradient has been employed is biomechanics. In contrast to hard tissues, soft tissues such as blood vessels and tendons can experience large deformations. An important contribution to their stress-modulated growth was recently made by Rodrigez *et al.* (1994). They introduced the decomposition of the corresponding deformation gradient into its elastic and growth parts, and applied it to evaluate the stress-dependent growth of an aorta. Subsequent efforts include the work of Taber and Eggers (1996), Taber and Perucchio (2000), Chen and Hoger (2000), Klisch *et al.* (2001), Lubarda and Hoger (2001), and Hoger *et al.* (2001). An analysis of the stress-modulated growth of isotropic pseudo-elastic soft tissues is presented in Section 4. The rate-type theory is constructed which incorporates an appealing structure of the evolution equation for the growth-induced stretch ratio. The concluding remarks are given in Section 5.

2. THERMOELASTICITY

In the constitutive theory of thermoelastic material response the intermediate configuration $\mathcal{B}_\theta$ is introduced by isothermal elastic destressing of the current configuration $\mathcal{B}$ (Fig. 1). If the isothermal elastic deformation gradient from $\mathcal{B}_\theta$ to $\mathcal{B}$ is $\mathbf{F}_e$, and the thermal deformation gradient from $\mathcal{B}_o$ to $\mathcal{B}_\theta$ is $\mathbf{F}_\theta$, the total deformation gradient $\mathbf{F}$ is

decomposed as

$$\mathbf{F} = \mathbf{F}_{\mathrm{e}} \cdot \mathbf{F}_{\theta} \,. \tag{2.1}$$

This decomposition was introduced in finite-strain thermoelasticity by Stojanović *et al.* (*op. cit*), and further employed by Stojanović (1972) and Mićunovic (1974). For inhomogeneous deformation and temperature fields only $\mathbf{F}$ is a true deformation gradient. The mappings from $\mathcal{B}_{\theta}$ to $\mathcal{B}$ and from $\mathcal{B}_{\mathrm{o}}$ to $\mathcal{B}_{\theta}$, on the other hand, are generally not continuous one-to-one mappings, so that $\mathbf{F}_{\mathrm{e}}$ and $\mathbf{F}_{\theta}$ are defined as the point functions or local deformation gradients. The decomposition (2.1) is not unique because arbitrary rigid-body rotation can be superposed to $\mathcal{B}_{\theta}$ preserving it unstressed. However, we shall specify $\mathbf{F}_{\theta}$ uniquely in each considered case, depending on the type of material anisotropy. For example, for an orthotropic material with the principal axes of orthotropy parallel to unit vectors $\mathbf{m}^{\mathrm{o}}$, $\mathbf{n}^{\mathrm{o}}$, and $\mathbf{m}^{\mathrm{o}} \times \mathbf{n}^{\mathrm{o}}$ in the configuration $\mathcal{B}_{\mathrm{o}}$, we specify $\mathbf{F}_{\theta}$ by

$$\mathbf{F}_{\theta} = \vartheta \mathbf{I} + (\beta - \vartheta)\, \mathbf{m}^{\mathrm{o}} \otimes \mathbf{m}^{\mathrm{o}} + (\gamma - \vartheta)\, \mathbf{n}^{\mathrm{o}} \otimes \mathbf{n}^{\mathrm{o}} \,. \tag{2.2}$$

The stretch ratios due to thermal expansion in the directions $\mathbf{m}^{\mathrm{o}}$ and $\mathbf{n}^{\mathrm{o}}$ are $\beta = \beta(\theta)$ and $\gamma = \gamma(\theta)$, while $\vartheta = \vartheta(\theta)$ is the stretch ratio in the direction $\mathbf{m}^{\mathrm{o}} \times \mathbf{n}^{\mathrm{o}}$. The second-order unit tensor is denoted by $\mathbf{I}$. A modification of the representation (2.2) to transversely isotropic materials is straightforward. The elastic Lagrangian strain and its rate can be expressed as

$$\mathbf{E}_{\mathrm{e}} = \mathbf{F}_{\theta}^{-T} \cdot (\mathbf{E} - \mathbf{E}_{\theta}) \cdot \mathbf{F}_{\theta}^{-1} \,, \tag{2.3}$$

$$\dot{\mathbf{E}}_{\mathrm{e}} = \mathbf{F}_{\theta}^{-T} \cdot \dot{\mathbf{E}} \cdot \mathbf{F}_{\theta}^{-1} - \mathbf{D}_{\theta} - \mathbf{E}_{\mathrm{e}} \cdot \mathbf{L}_{\theta} - \mathbf{L}_{\theta}^{T} \cdot \mathbf{E}_{\mathrm{e}} \,, \tag{2.4}$$

where $\mathbf{L}_{\theta} = \dot{\mathbf{F}}_{\theta} \cdot \mathbf{F}_{\theta}^{-1}$, and $\mathbf{D}_{\theta}$ is its symmetric part. The elastic and thermal strains are defined by

$$\mathbf{E}_{\mathrm{e}} = \frac{1}{2} \left(\mathbf{F}_{\mathrm{e}}^{T} \cdot \mathbf{F}_{\mathrm{e}} - \mathbf{I} \right) , \quad \mathbf{E}_{\theta} = \frac{1}{2} \left(\mathbf{F}_{\theta}^{T} \cdot \mathbf{F}_{\theta} - \mathbf{I} \right) . \tag{2.5}$$

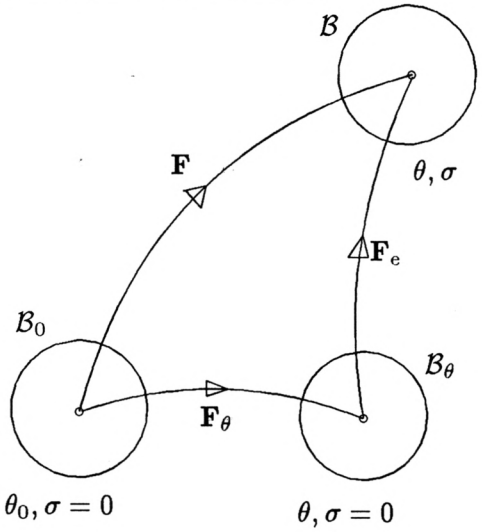

Figure 1: The intermediate configuration $\mathcal{B}_\theta$ at nonuniform temperature θ is obtained from the deformed configuration $\mathcal{B}$ by isothermal destressing to zero stress. The deformation gradient from initial to deformed configuration $\mathbf{F}$ is decomposed into elastic part $\mathbf{F}_e$ and thermal part $\mathbf{F}_\theta$, such that $\mathbf{F} = \mathbf{F}_e \cdot \mathbf{F}_\theta$.

The analysis will be restricted in the sequel to isotropic materials, for which the thermal part of the deformation gradient is

$$\mathbf{F}_\theta = \vartheta(\theta)\,\mathbf{I}\,. \tag{2.6}$$

The scalar $\vartheta = \vartheta(\theta)$ is the thermal stretch ratio in any material direction. In this case

$$\mathbf{E}_e = \frac{1}{\vartheta^2}\,(\mathbf{E} - \mathbf{E}_\theta)\,, \quad \mathbf{E}_\theta = \frac{1}{2}\,(\vartheta^2 - 1)\,\mathbf{I}\,. \tag{2.7}$$

It is noted that

$$\mathbf{I} + 2\mathbf{E} = \vartheta^2(\mathbf{I} + 2\mathbf{E}_e)\,. \tag{2.8}$$

Since the differential connection between the thermal stretch ratio and the coefficient of thermal expansion is

$$\alpha(\theta) = \frac{1}{\vartheta}\,\frac{\mathrm{d}\vartheta}{\mathrm{d}\theta}\,, \tag{2.9}$$

the rate of elastic strain can be written as

$$\dot{\mathbf{E}}_{\mathrm{e}} = \frac{1}{\vartheta^2(\theta)} \left[\dot{\mathbf{E}} - \alpha(\theta)(\mathbf{I} + 2\mathbf{E})\dot{\theta} \right]. \tag{2.10}$$

2.1 Stress Response

Within the model of the multiplicative decomposition, the Helmholtz free energy can be split into two parts, such that

$$\psi = \psi_{\mathrm{e}}(\mathbf{E}_{\mathrm{e}}, \theta) + \psi_{\theta}(\theta), \tag{2.11}$$

where ψ_{e} is an isotropic function of the elastic strain $\mathbf{E}_{\mathrm{e}}$ and temperature θ. The time-rate of the free energy is

$$\dot{\psi} = \frac{\partial \psi_{\mathrm{e}}}{\partial \mathbf{E}_{\mathrm{e}}} : \dot{\mathbf{E}}_{\mathrm{e}} + \frac{\partial \psi_{\mathrm{e}}}{\partial \theta}\dot{\theta} + \frac{\mathrm{d}\psi_{\theta}}{\mathrm{d}\theta}\dot{\theta}. \tag{2.12}$$

Upon substitution of Eq. (2.10), there follows

$$\dot{\psi} = \frac{1}{\vartheta^2}\frac{\partial \psi_{\mathrm{e}}}{\partial \mathbf{E}_{\mathrm{e}}} : \dot{\mathbf{E}} - \left[\frac{\alpha}{\vartheta^2}\frac{\partial \psi_{\mathrm{e}}}{\partial \mathbf{E}_{\mathrm{e}}} : (\mathbf{I} + 2\mathbf{E}) - \frac{\partial \psi_{\mathrm{e}}}{\partial \theta} - \frac{\mathrm{d}\psi_{\theta}}{\mathrm{d}\theta} \right] \dot{\theta}. \tag{2.13}$$

The comparison with the energy equation

$$\dot{\psi} = \frac{1}{\rho_{\mathrm{o}}} \mathbf{T} : \dot{\mathbf{E}} - \eta\dot{\theta} \tag{2.14}$$

establishes the constitutive relations

$$\mathbf{T} = \frac{\rho_{\mathrm{o}}}{\vartheta^2}\frac{\partial \psi_{\mathrm{e}}}{\partial \mathbf{E}_{\mathrm{e}}}, \tag{2.15}$$

$$\eta = \alpha \frac{\partial \psi_{\mathrm{e}}}{\partial \mathbf{E}_{\mathrm{e}}} : (\mathbf{I} + 2\mathbf{E}_{\mathrm{e}}) - \frac{\partial \psi_{\mathrm{e}}}{\partial \theta} - \frac{\mathrm{d}\psi_{\theta}}{\mathrm{d}\theta}. \tag{2.16}$$

In view of the relationship $\rho_{\mathrm{o}} = \vartheta^3 \rho_{\theta}$ between the densities ρ_{o} in the configuration $\mathcal{B}_{\mathrm{o}}$ and ρ_{θ} in the configuration $\mathcal{B}_{\theta}$, the stress response in Eq. (2.15) can also be written as

$$\mathbf{T} = \vartheta\, \mathbf{T}_{\mathrm{e}}, \quad \mathbf{T}_{\mathrm{e}} = \rho_{\theta}\frac{\partial \psi_{\mathrm{e}}}{\partial \mathbf{E}_{\mathrm{e}}}. \tag{2.17}$$

An appealing feature of the thermoelastic constitutive formulation based on the multiplicative decomposition is that the function $\psi_e(\mathbf{E}_e, \theta)$ can be taken as one of the well-known strain energy functions of isothermal finite-strain elasticity, except that the coefficients of the strain-dependent terms are functions of the temperature (Vujošević and Lubarda, 2002). For example, suppose that ψ_e is a quadratic function of the elastic strain components, such that

$$\rho_\theta \psi_e = \frac{1}{2}\lambda(\theta)(\operatorname{tr}\mathbf{E}_e)^2 + \mu(\theta)\,\mathbf{E}_e : \mathbf{E}_e\,, \tag{2.18}$$

where $\lambda(\theta)$ and $\mu(\theta)$ are the temperature-dependent Lamé moduli. It follows that

$$\mathbf{T}_e = \mathbf{\Lambda}_e(\theta) : \mathbf{E}_e\,, \quad \mathbf{\Lambda}_e(\theta) = \lambda(\theta)\,\mathbf{I}\otimes\mathbf{I} + 2\mu(\theta)\,\mathbf{II}\,. \tag{2.19}$$

The temperature-dependent elastic moduli tensor is $\mathbf{\Lambda}_e(\theta)$, while $\mathbf{II}$ stands for the fourth-order unit tensor. Consequently, substituting Eqs. (2.10) and (2.19) into $\mathbf{T} = \vartheta\,\mathbf{T}_e$, the stress response becomes

$$\mathbf{T} = \frac{1}{\vartheta(\theta)}\Big[\lambda(\theta)(\operatorname{tr}\mathbf{E})\,\mathbf{I} + 2\mu(\theta)\,\mathbf{E}\Big] - \frac{3}{2}\left[\vartheta(\theta) - \frac{1}{\vartheta(\theta)}\right]\kappa(\theta)\,\mathbf{I}\,. \tag{2.20}$$

The temperature-dependent bulk modulus is $\kappa(\theta)$. This is an exact expression for the thermoelastic stress response associated with the quadratic representation of ψ_e in terms of the finite elastic strain $\mathbf{E}_e$. If the Lamé moduli are taken to be temperature-independent, and if the approximation $\vartheta(\theta) \approx 1 + \alpha_o(\theta - \theta_o)$ is made (α_o being the coefficient of linear thermal expansion at $\theta = \theta_o$), Eq. (2.20) reduces to

$$\mathbf{T} = \lambda_o\,(\operatorname{tr}\mathbf{E})\,\mathbf{I} + 2\mu_o\,\mathbf{E} - 3\alpha_o(\theta - \theta_o)\kappa_o\,\mathbf{I}\,. \tag{2.21}$$

When $\mathbf{E}$ and $\mathbf{T}$ are interpreted as the infinitesimal strain and the Cauchy stress, the equation coincides with the well-known Duhamel–Neumann expression of isotropic linear thermoelasticity (e.g., Boley and Weiner, 1960; Parkus, 1968; Nowacki, 1986).

2.2 Entropy Expression

In the case of the quadratic strain energy representation (2.18), we have $\rho_o \psi_e = \vartheta^3 \mathbf{T}_e : \mathbf{E}_e/2$, so that

$$\rho_o \left(\frac{\partial \psi_e}{\partial \theta}\right)_{\mathbf{E}_e} = \frac{3}{2}\vartheta^2 \frac{\mathrm{d}\vartheta}{\mathrm{d}\theta}\mathbf{T}_e : \mathbf{E}_e + \frac{1}{2}\vartheta^3 \left(\frac{\partial \mathbf{T}_e}{\partial \theta}\right)_{\mathbf{E}_e} : \mathbf{E}_e, \tag{2.22}$$

i.e.,

$$\rho_o \left(\frac{\partial \psi_e}{\partial \theta}\right)_{\mathbf{E}_e} = \frac{3}{2}\alpha \left[\mathbf{T} : \mathbf{E} - \frac{1}{2}(\vartheta^2 - 1)\mathrm{tr}\mathbf{T}\right] + \frac{1}{2}\vartheta^3 \left(\frac{\partial \mathbf{T}_e}{\partial \theta}\right)_{\mathbf{E}_e} : \mathbf{E}_e. \tag{2.23}$$

It can be readily verified that

$$\vartheta \left(\frac{\partial \mathbf{T}_e}{\partial \theta}\right)_{\mathbf{E}_e} = \left(\frac{\partial \mathbf{T}}{\partial \theta}\right)_{\mathbf{E}} + \alpha\,(\mathbf{T} + 3\,\vartheta\,\kappa\,\mathbf{I})\,, \tag{2.24}$$

and

$$\begin{aligned} \vartheta^3 \left(\tfrac{\partial \mathbf{T}_e}{\partial \theta}\right)_{\mathbf{E}_e} : \mathbf{E}_e \; &= \left(\tfrac{\partial \mathbf{T}}{\partial \theta}\right)_{\mathbf{E}} : \left[\mathbf{E} - \tfrac{1}{2}\,(\vartheta^2 - 1)\,\mathbf{I}\right] \\ &\quad + \alpha \left[\mathbf{T} : \mathbf{E} + \tfrac{1}{2}\,(1 + \vartheta^2)\,\mathrm{tr}\,\mathbf{T}\right]. \end{aligned} \tag{2.25}$$

Inserting Eq. (2.25) into Eq. (2.23) gives

$$\begin{aligned} \rho_o \left(\tfrac{\partial \psi_e}{\partial \theta}\right)_{\mathbf{E}_e} \; &= 2\,\alpha\,\mathbf{T} : \mathbf{E} + \tfrac{1}{2}\,\alpha\,(2 - \vartheta^2)\,\mathrm{tr}\,\mathbf{T} \\ &\quad + \tfrac{1}{2} \left(\tfrac{\partial \mathbf{T}}{\partial \theta}\right)_{\mathbf{E}} : \left[\mathbf{E} - \tfrac{1}{2}\,(\vartheta^2 - 1)\,\mathbf{I}\right]. \end{aligned} \tag{2.26}$$

When this is substituted into Eq. (2.16), the entropy becomes

$$\eta = \frac{1}{2\rho_o}\left[3\,\vartheta\,\alpha\,\kappa\,\mathbf{I} - \left(\frac{\partial \mathbf{T}}{\partial \theta}\right)_{\mathbf{E}}\right] : \left[\mathbf{E} - \frac{1}{2}\,(\vartheta^2 - 1)\,\mathbf{I}\right] - \frac{\mathrm{d}\psi_\theta}{\mathrm{d}\theta}. \tag{2.27}$$

Recalling the standard expression for the latent heat (e.g., Fung, 1965), we finally have

$$\eta = \frac{1}{2}\left(\frac{1}{\theta}\,\boldsymbol{\ell}_E + \frac{3}{\rho_o}\,\vartheta\,\alpha\,\kappa\,\mathbf{I}\right) : \left[\mathbf{E} - \frac{1}{2}\,(\vartheta^2 - 1)\,\mathbf{I}\right] - \frac{\mathrm{d}\psi_\theta}{\mathrm{d}\theta}. \tag{2.28}$$

This is an exact expression for η within the approximation used for the elastic strain energy. It was originally derived by Vujošević and

Lubarda, *op. cit.*. The second-order tensor of the latent heat $\boldsymbol{\ell}_E$ can be calculated from Eq. (2.24) as

$$\boldsymbol{\ell}_E = -\frac{1}{\rho_o}\theta\left(\frac{\partial \mathbf{T}}{\partial \theta}\right)_{\mathbf{E}} = -\frac{1}{\rho_o}\theta\left[\vartheta\left(\frac{\partial \mathbf{T}_e}{\partial \theta}\right)_{\mathbf{E}_e} - \alpha\,(\mathbf{T} + 3\,\vartheta\,\kappa\,\mathbf{I})\right], \quad (2.29)$$

which gives

$$\boldsymbol{\ell}_E = \frac{1}{\rho_o}\theta\left\{\alpha\,(\mathbf{T} + 3\,\vartheta\,\kappa\,\mathbf{I}) - \frac{1}{\vartheta}\frac{\mathrm{d}\boldsymbol{\Lambda}_e}{\mathrm{d}\theta} : \left[\mathbf{E} - \frac{1}{2}\,(\vartheta^2 - 1)\,\mathbf{I}\right]\right\}. \quad (2.30)$$

If the elastic moduli are independent of temperature, and if the stress components are much smaller that the bulk modulus, the specific heat becomes $\boldsymbol{\ell}_E = 3\,\vartheta\,\alpha\,\theta\,\kappa\,\mathbf{I}/\rho_o$, while the entropy expression (2.28) reduces to

$$\eta = \frac{3}{\rho_o}\vartheta\,\alpha\,\kappa\left[\operatorname{tr}\mathbf{E} - \frac{3}{2}\,(\vartheta^2 - 1)\right] - \frac{\mathrm{d}\psi_\theta}{\mathrm{d}\theta}. \quad (2.31)$$

The function ψ_θ can be selected according to experimental data for the specific heat c_E. For example, if

$$\psi_\theta = -\frac{1}{2}\left(\frac{c_E^o}{\theta_o} + \frac{9}{\rho_o}\alpha_o^2\,\kappa_o\right)(\theta - \theta_o)^2, \quad (2.32)$$

equation (2.31) becomes

$$\eta = \frac{3}{\rho_o}\alpha_o\,\kappa_o\operatorname{tr}\mathbf{E} + \frac{c_E^o}{\theta_o}\,(\theta - \theta_o), \quad (2.33)$$

which is in agreement with the result of the linearized classical theory of thermoelasticity.

2. ELASTOPLASTICITY

The intermediate configuration in finite-deformation elastoplasticity, obtained from the current configuration by elastic destressing to zero stress (Fig. 2), differs from the initial configuration by a residual

(plastic) deformation, and from the current configuration by a reversible (elastic) deformation. The corresponding multiplicative decomposition of the elastoplastic deformation gradient into its elastic and plastic part, introduced by Lee (1969), reads

$$\mathbf{F} = \mathbf{F}_e \cdot \mathbf{F}_p \,. \tag{3.1}$$

In the case when elastic destressing to zero stress is not physically achievable due to onset of reverse plastic deformation before the state of zero stress is reached, the intermediate configuration can be conceptually introduced by virtual destressing to zero stress, locking inelastic structural changes that would occur during the actual destressing. The deformation gradients $\mathbf{F}_e$ and $\mathbf{F}_p$ are not uniquely defined because intermediate unstressed configuration is not unique; arbitrary local material rotations can be superposed to intermediate configuration preserving it unstressed. In the applications, however, the decomposition can be made unique by additional specifications dictated by the nature of the considered material model. For example, for elastically isotropic materials the stress response from $\mathcal{B}_p$ to $\mathcal{B}$ does not depend on the rotation $\mathbf{R}_e$ from the polar decomposition $\mathbf{F}_e = \mathbf{V}_e \cdot \mathbf{R}_e$. Consequently, the intermediate configuration can in this case be specified uniquely by requiring that elastic unloading takes place without rotation.

By introducing the multiplicative decomposition of the deformation gradient (3.1), the velocity gradient becomes

$$\mathbf{L} = \dot{\mathbf{F}}_e \cdot \mathbf{F}_e^{-1} + \mathbf{F}_e \cdot \left(\dot{\mathbf{F}}_p \cdot \mathbf{F}_p^{-1}\right) \cdot \mathbf{F}_e^{-1} \,. \tag{3.2}$$

The rate of deformation $\mathbf{D}$ and the spin $\mathbf{W}$ are given by its symmetric and antisymmetric part, i.e.,

$$\mathbf{D} = \left(\dot{\mathbf{F}}_e \cdot \mathbf{F}_e^{-1}\right)_s + \left[\mathbf{F}_e \cdot \left(\dot{\mathbf{F}}_p \cdot \mathbf{F}_p^{-1}\right) \cdot \mathbf{F}_e^{-1}\right]_s , \tag{3.3}$$

$$\mathbf{W} = \left(\dot{\mathbf{F}}_e \cdot \mathbf{F}_e^{-1}\right)_a + \left[\mathbf{F}_e \cdot \left(\dot{\mathbf{F}}_p \cdot \mathbf{F}_p^{-1}\right) \cdot \mathbf{F}_e^{-1}\right]_a . \tag{3.4}$$

For later purposes, it is convenient to identify the spin tensor

$$\boldsymbol{\omega}_p = \left[\mathbf{F}_e \cdot \left(\dot{\mathbf{F}}_p \cdot \mathbf{F}_p^{-1}\right) \cdot \mathbf{F}_e^{-1}\right]_a . \tag{3.5}$$

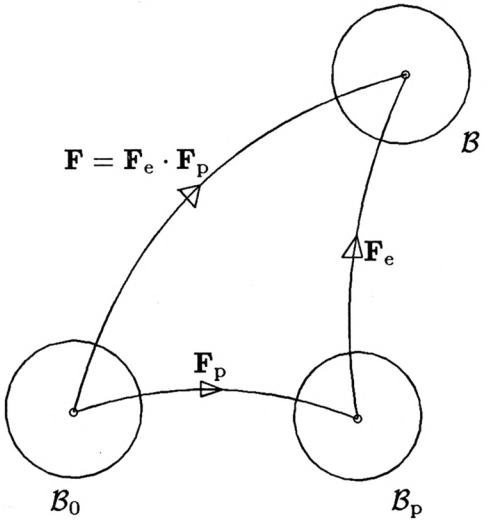

Figure 2: The intermediate configuration $\mathcal{B}_p$ is obtained from the deformed configuration $\mathcal{B}$ by destressing to zero stress. The elastoplastic deformation gradient is decomposed into elastic and plastic parts, such that $\mathbf{F} = \mathbf{F}_e \cdot \mathbf{F}_p$.

3.3 Partition of Elastoplastic Rate of Deformation

Suppose that material is elastically isotropic in its initial configuration, and that plastic deformation does not affect its elastic properties. The elastic response is then independent of any rotation superposed to the intermediate configuration, and is given by

$$\boldsymbol{\tau} = \mathbf{F}_e \cdot \frac{\partial \Psi_e(\mathbf{E}_e)}{\partial \mathbf{E}_e} \cdot \mathbf{F}_e^T . \tag{3.6}$$

The elastic strain energy per unit unstressed volume, $\Psi_e = \rho_o \psi_e$, is an isotropic function of the Lagrangian strain $\mathbf{E}_e$. Plastic deformation is assumed to be incompressible ($\det \mathbf{F}_e = \det \mathbf{F}$), so that $\boldsymbol{\tau} = (\det \mathbf{F})\boldsymbol{\sigma}$ is the Kirchhoff stress (the Cauchy stress $\boldsymbol{\sigma}$ weighted by $\det \mathbf{F}$). By

differentiating Eq. (3.6), we obtain

$$\dot{\boldsymbol{\tau}} - \left(\dot{\mathbf{F}}_{\mathrm{e}} \cdot \mathbf{F}_{\mathrm{e}}^{-1}\right) \cdot \boldsymbol{\tau} - \boldsymbol{\tau} \cdot \left(\dot{\mathbf{F}}_{\mathrm{e}} \cdot \mathbf{F}_{\mathrm{e}}^{-1}\right)^{T} = \bar{\boldsymbol{\mathcal{L}}} : \left(\dot{\mathbf{F}}_{\mathrm{e}} \cdot \mathbf{F}_{\mathrm{e}}^{-1}\right)_{\mathrm{s}} . \tag{3.7}$$

The rectangular components of $\bar{\boldsymbol{\mathcal{L}}}$ are

$$\bar{\mathcal{L}}_{ijkl} = F^{\mathrm{e}}_{im} F^{\mathrm{e}}_{jn} \frac{\partial^2 \Psi_{\mathrm{e}}}{\partial E^{\mathrm{e}}_{mn} \partial E^{\mathrm{e}}_{pq}} F^{\mathrm{e}}_{kp} F^{\mathrm{e}}_{lq} . \tag{3.8}$$

Equation (3.7) can be rewritten as

$$\dot{\boldsymbol{\tau}} - \left(\dot{\mathbf{F}}_{\mathrm{e}} \cdot \mathbf{F}_{\mathrm{e}}^{-1}\right)_{\mathrm{a}} \cdot \boldsymbol{\tau} + \boldsymbol{\tau} \cdot \left(\dot{\mathbf{F}}_{\mathrm{e}} \cdot \mathbf{F}_{\mathrm{e}}^{-1}\right)_{\mathrm{a}} = \boldsymbol{\mathcal{L}} : \left(\dot{\mathbf{F}}_{\mathrm{e}} \cdot \mathbf{F}_{\mathrm{e}}^{-1}\right)_{\mathrm{s}} , \tag{3.9}$$

where the new instantaneous moduli are given by

$$\mathcal{L}_{ijkl} = \bar{\mathcal{L}}_{ijkl} + \frac{1}{2} \left(\tau_{ik}\delta_{jl} + \tau_{jk}\delta_{il} + \tau_{il}\delta_{jk} + \tau_{jl}\delta_{ik}\right) . \tag{3.10}$$

The elastic deformation gradient $\mathbf{F}_{\mathrm{e}}$ is defined relative to intermediate configuration which changes during elastoplastic deformation. This causes two difficulties in the identification of elastic rate of deformation $\mathbf{D}_{\mathrm{e}}$ (Lubarda and Shih, 1994). First, since $\mathbf{F}_{\mathrm{e}}$ and $\mathbf{F}_{\mathrm{p}}$ are specified only to within an arbitrary rotation, the velocity gradient $\dot{\mathbf{F}}_{\mathrm{e}} \cdot \mathbf{F}_{\mathrm{e}}^{-1}$ and its symmetric and antisymmetric parts are not unique. Second, the deforming intermediate configuration also contributes to the elastic rate of deformation, which is not in general given only by $\left(\dot{\mathbf{F}}_{\mathrm{e}} \cdot \mathbf{F}_{\mathrm{e}}^{-1}\right)_{\mathrm{s}}$. To overcome these difficulties, a kinetic definition of elastic strain increment is used according to which $\mathbf{D}_{\mathrm{e}}\,\mathrm{d}t$ is a reversible part of the total strain increment $\mathbf{D}\,\mathrm{d}t$, recovered upon loading-unloading cycle of the Jaumann stress increment $\overset{\circ}{\boldsymbol{\tau}}\,\mathrm{d}t$. Thus,

$$\mathbf{D}_{\mathrm{e}} = \boldsymbol{\mathcal{L}}^{-1} : \overset{\circ}{\boldsymbol{\tau}}, \quad \overset{\circ}{\boldsymbol{\tau}} = \dot{\boldsymbol{\tau}} - \mathbf{W} \cdot \boldsymbol{\tau} + \boldsymbol{\tau} \cdot \mathbf{W} . \tag{3.11}$$

The remaining part of the total rate of deformation,

$$\mathbf{D}_{\mathrm{p}} = \mathbf{D} - \mathbf{D}_{\mathrm{e}} , \tag{3.12}$$

is the plastic part, which gives the residual strain increment left upon the considered infinitesimal cycle of stress. If the material obeys the

Ilyushin's postulate, so defined plastic rate of deformation is codirectional with the outward normal to a locally smooth yield surface in the Cauchy stress space (Hill, 1978).

Therefore, to identify in Eq. (3.9) the elastic strain rate in accord with the kinetic definition (3.11), we eliminate $\left(\dot{\mathbf{F}}_{\mathrm{e}} \cdot \mathbf{F}_{\mathrm{e}}^{-1}\right)_{\mathrm{a}}$ in terms of $\mathbf{W}$ and $\boldsymbol{\omega}_{\mathrm{p}}$ and obtain

$$\overset{\circ}{\boldsymbol{\tau}} = \boldsymbol{\mathcal{L}} : \left(\dot{\mathbf{F}}_{\mathrm{e}} \cdot \mathbf{F}_{\mathrm{e}}^{-1}\right)_{\mathrm{s}} - \boldsymbol{\omega}_{\mathrm{p}} \cdot \boldsymbol{\tau} + \boldsymbol{\tau} \cdot \boldsymbol{\omega}_{\mathrm{p}} \,. \tag{3.13}$$

Consequently, the elastic rate of deformation is given by

$$\mathbf{D}_{\mathrm{e}} = \left(\dot{\mathbf{F}}_{\mathrm{e}} \cdot \mathbf{F}_{\mathrm{e}}^{-1}\right)_{\mathrm{s}} - \boldsymbol{\mathcal{L}}^{-1} : (\boldsymbol{\omega}_{\mathrm{p}} \cdot \boldsymbol{\tau} - \boldsymbol{\tau} \cdot \boldsymbol{\omega}_{\mathrm{p}}) \,. \tag{3.14}$$

The corresponding plastic rate of deformation is

$$\mathbf{D}_{\mathrm{p}} = \left[\mathbf{F}_{\mathrm{e}} \cdot \left(\dot{\mathbf{F}}_{\mathrm{p}} \cdot \mathbf{F}_{\mathrm{p}}^{-1}\right) \cdot \mathbf{F}_{\mathrm{e}}^{-1}\right]_{\mathrm{s}} + \boldsymbol{\mathcal{L}}^{-1} : (\boldsymbol{\omega}_{\mathrm{p}} \cdot \boldsymbol{\tau} - \boldsymbol{\tau} \cdot \boldsymbol{\omega}_{\mathrm{p}}) \,. \tag{3.15}$$

Since $\boldsymbol{\mathcal{L}}^{-1}$ and $\overset{\circ}{\boldsymbol{\tau}}$ in (3.11) are independent of the superposed rotation to the intermediate configuration, Eq. (3.14) specifies $\mathbf{D}_{\mathrm{e}}$ uniquely. In contrast, its constituents, $\left(\dot{\mathbf{F}}_{\mathrm{e}} \cdot \mathbf{F}_{\mathrm{e}}^{-1}\right)_{\mathrm{s}}$ and the term associated with the spin $\boldsymbol{\omega}_{\mathrm{p}}$, do depend on the choice of the intermediate configuration. Similar remarks apply to plastic rate of deformation $\mathbf{D}_{\mathrm{p}}$ in its representation (3.15).

The right hand side of (3.14) is in general the correct expression for the elastic rate of deformation, and not $\left(\dot{\mathbf{F}}_{\mathrm{e}} \cdot \mathbf{F}_{\mathrm{e}}^{-1}\right)_{\mathrm{s}}$ alone. Only if the intermediate configuration (i.e., rotation $\mathbf{R}_{\mathrm{e}}$ during destressing program) is chosen such that the spin $\boldsymbol{\omega}_{\mathrm{p}} = \mathbf{0}$, the rate of deformation $\left(\dot{\mathbf{F}}_{\mathrm{e}} \cdot \mathbf{F}_{\mathrm{e}}^{-1}\right)_{\mathrm{s}}$ is exactly equal to $\mathbf{D}_{\mathrm{e}}$. Within the framework under discussion, this choice of the spin represents a geometric (kinematic) specification of the intermediate configuration. It is not a constitutive assumption and has no consequences on (3.14). We could just as well define an intermediate configuration by requiring that the spin $\left(\dot{\mathbf{F}}_{\mathrm{e}} \cdot \mathbf{F}_{\mathrm{e}}^{-1}\right)_{\mathrm{a}}$ vanishes identically. In this case, $\boldsymbol{\omega}_{\mathrm{p}} = \mathbf{W}$. The end result is still equation (3.14), as can be checked by inspection. The described partition of $\mathbf{D}$ into its elastic and plastic parts within the framework of

the multiplicative decomposition has been a topic of active research and debate for number of years; Freund (1970), Kratochvil (1973), Lubarda and Lee (1981), Nemat-Nasser (1982), Simo and Ortiz (1985), Loret (1983), Dafalias (1985, 1987), Lubarda and Shih (1994). The constitutive structure for the plastic part of the rate of deformation tensor is constructed by using the concept of the yield surface. This gives

$$\mathbf{D}_{\mathrm{p}} = \frac{1}{h}\left(\frac{\partial f}{\partial \boldsymbol{\sigma}} \otimes \frac{\partial f}{\partial \boldsymbol{\sigma}}\right) : \overset{\circ}{\boldsymbol{\tau}}, \tag{3.16}$$

where h is a scalar parameter of the deformation history, and $f = 0$ is the yield surface (e.g., Naghdi, 1990; Simo and Hughes, 1998). Details can be found in Lubarda (2002).

3.4 Analysis of Elastic Rate of Deformation

The elastic rate of deformation of elastically isotropic material can be expressed as

$$\mathbf{D}_{\mathrm{e}} = \left(\overset{\bullet}{\mathbf{F}}_{\mathrm{e}} \cdot \mathbf{F}_{\mathrm{e}}^{-1}\right)_{\mathrm{s}} = \left(\dot{\mathbf{F}}_{\mathrm{e}} \cdot \mathbf{F}_{\mathrm{e}}^{-1}\right)_{\mathrm{s}} + \left(\mathbf{F}_{\mathrm{e}} \cdot \boldsymbol{\Omega}_{\mathrm{p}} \cdot \mathbf{F}_{\mathrm{e}}^{-1}\right)_{\mathrm{s}}. \tag{3.17}$$

The Jaumann derivative of $\mathbf{F}_{\mathrm{e}}$ is here defined by

$$\overset{\bullet}{\mathbf{F}}_{\mathrm{e}} = \dot{\mathbf{F}}_{\mathrm{e}} - \boldsymbol{\Omega}_{\mathrm{p}} \cdot \mathbf{F}_{\mathrm{e}} + \mathbf{F}_{\mathrm{e}} \cdot \boldsymbol{\Omega}_{\mathrm{p}}, \tag{3.18}$$

which represents the rate of $\mathbf{F}_{\mathrm{e}}$ observed in the coordinate systems that rotate with the spin $\boldsymbol{\Omega}_{\mathrm{p}}$ in both, current and intermediate configurations. The spin $\boldsymbol{\Omega}_{\mathrm{p}}$ is defined as the solution of the matrix equation

$$\left(\dot{\mathbf{F}}_{\mathrm{e}} \cdot \mathbf{F}_{\mathrm{e}}^{-1}\right)_{\mathrm{a}} + \left(\mathbf{F}_{\mathrm{e}} \cdot \boldsymbol{\Omega}_{\mathrm{p}} \cdot \mathbf{F}_{\mathrm{e}}^{-1}\right)_{\mathrm{a}} = \mathbf{W}. \tag{3.19}$$

The proof proceeds by applying the Jaumann derivative with respect to spin $\boldsymbol{\Omega}_{\mathrm{p}}$ to both sides of Eq. (3.6), which gives

$$\overset{\bullet}{\boldsymbol{\tau}} = \left(\overset{\bullet}{\mathbf{F}}_{\mathrm{e}} \cdot \mathbf{F}_{\mathrm{e}}^{-1}\right) \cdot \boldsymbol{\tau} + \boldsymbol{\tau} \cdot \left(\overset{\bullet}{\mathbf{F}}_{\mathrm{e}} \cdot \mathbf{F}_{\mathrm{e}}^{-1}\right)^{T} + \mathbf{F}_{\mathrm{e}} \cdot \left(\frac{\partial^2 \Psi_{\mathrm{e}}}{\partial \mathbf{E}_{\mathrm{e}} \otimes \partial \mathbf{E}_{\mathrm{e}}} : \overset{\bullet}{\mathbf{E}}_{\mathrm{e}}\right) \cdot \mathbf{F}_{\mathrm{e}}^{T}. \tag{3.20}$$

Since

$$\overset{\bullet}{\mathbf{F}}_{\mathrm{e}} \cdot \mathbf{F}_{\mathrm{e}}^{-1} = \mathbf{D}_{\mathrm{e}} + \mathbf{W} - \boldsymbol{\Omega}_{\mathrm{p}}\,, \tag{3.21}$$

the substitution into Eq. (3.20) yields

$$\overset{\circ}{\boldsymbol{\tau}} = \boldsymbol{\mathcal{L}} : \mathbf{D}_{\mathrm{e}}, \quad \mathbf{D}_{\mathrm{e}} = \left(\overset{\bullet}{\mathbf{F}}_{\mathrm{e}} \cdot \mathbf{F}_{\mathrm{e}}^{-1}\right)_{\mathrm{s}}. \tag{3.22}$$

The two contributions to the elastic rate of deformation $\mathbf{D}_{\mathrm{e}}$ in Eq. (3.17) both depend on the choice of intermediate configuration, i.e., on the elastic rotation $\mathbf{R}_{\mathrm{e}}$ of the destressing program, but their sum giving $\mathbf{D}_{\mathrm{e}}$ does not. If elastic destressing is performed without rotation ($\mathbf{R}_{\mathrm{e}} = \mathbf{I}$), the spin $\boldsymbol{\Omega}_{\mathrm{p}} = \boldsymbol{\Omega}_{\mathrm{p}}^{\circ}$ is the solution of

$$\left(\dot{\mathbf{V}}_{\mathrm{e}} \cdot \mathbf{V}_{\mathrm{e}}^{-1}\right)_{\mathrm{a}} + \left(\mathbf{V}_{\mathrm{e}} \cdot \boldsymbol{\Omega}_{\mathrm{p}}^{\circ} \cdot \mathbf{V}_{\mathrm{e}}^{-1}\right)_{\mathrm{a}} = \mathbf{W}\,. \tag{3.23}$$

This defines uniquely the spin $\boldsymbol{\Omega}_{\mathrm{p}}^{\circ}$ in terms of $\mathbf{W}$, $\mathbf{V}_{\mathrm{e}}$ and $\dot{\mathbf{V}}_{\mathrm{e}}$. The elastic rate of deformation (3.17) is in this case

$$\mathbf{D}_{\mathrm{e}} = \left(\overset{\bullet}{\mathbf{V}}_{\mathrm{e}} \cdot \mathbf{V}_{\mathrm{e}}^{-1}\right)_{\mathrm{s}} = \left(\dot{\mathbf{V}}_{\mathrm{e}} \cdot \mathbf{V}_{\mathrm{e}}^{-1}\right)_{\mathrm{s}} + \left(\mathbf{V}_{\mathrm{e}} \cdot \boldsymbol{\Omega}_{\mathrm{p}}^{\circ} \cdot \mathbf{V}_{\mathrm{e}}^{-1}\right)_{\mathrm{s}}. \tag{3.24}$$

The first term on the right-hand side represents the contribution to $\mathbf{D}_{\mathrm{e}}$ from elastic stretching rate $\left(\dot{\mathbf{V}}_{\mathrm{e}} \cdot \mathbf{V}_{\mathrm{e}}^{-1}\right)_{\mathrm{s}}$, while the second term depends on the spin $\boldsymbol{\Omega}_{\mathrm{p}}^{\circ}$ and accounts for the effects of deforming and rotating intermediate configuration (Lubarda, 1991).

The representation of the elastic rate of deformation in Eq. (3.17) involves only kinematic quantities ($\mathbf{F}_{\mathrm{e}}$ and $\boldsymbol{\Omega}_{\mathrm{p}}$), while the representation (3.14) involves both kinematic and kinetic quantities. Clearly,

$$\left(\mathbf{F}_{\mathrm{e}} \cdot \boldsymbol{\Omega}_{\mathrm{p}} \cdot \mathbf{F}_{\mathrm{e}}^{-1}\right)_{\mathrm{s}} = -\boldsymbol{\mathcal{L}}^{-1} : (\boldsymbol{\omega}_{\mathrm{p}} \cdot \boldsymbol{\tau} - \boldsymbol{\tau} \cdot \boldsymbol{\omega}_{\mathrm{p}})\,. \tag{3.25}$$

Note also that the elastic strain expression (3.17) can be recast in the form

$$\mathbf{D}_{\mathrm{e}} = \frac{1}{2}\mathbf{F}_{\mathrm{e}}^{-T} \cdot \overset{\bullet}{\mathbf{C}}_{\mathrm{e}} \cdot \mathbf{F}_{\mathrm{e}}^{-1}, \quad \overset{\bullet}{\mathbf{C}}_{\mathrm{e}} = \dot{\mathbf{C}}_{\mathrm{e}} - \boldsymbol{\Omega}_{\mathrm{p}} \cdot \mathbf{C}_{\mathrm{e}} + \mathbf{C}_{\mathrm{e}} \cdot \boldsymbol{\Omega}_{\mathrm{p}}\,. \tag{3.26}$$

This expression, as well as (3.17), holds for the elastoplastic deformation of elastically isotropic materials, regardless of whether the material hardens isotropically or anisotropically.

3.5 Crystal Plasticity

In single crystals in which crystallographic slip is assumed to be the only mechanism of plastic deformation, the material flows through the lattice via dislocation motion, while the lattice itself, with the material embedded to it, undergoes elastic deformation and rotation. The discrete dislocation substructure is ignored, and plastic deformation is considered to occur in the form of smooth shearing on the slip planes and in the slip directions. Such continuum slip model from the pioneering work of Taylor (1938) was employed in the analysis by Hill and Rice (1972), Mandel (1974), Asaro and Rice (1977), Hill and Havner (1982), Lubarda (1999), Lubarda and Benson (2001), and others. The deformation gradient is decomposed as

$$\mathbf{F} = \mathbf{F}_* \cdot \mathbf{F}_\mathrm{p}\,, \tag{3.27}$$

where $\mathbf{F}_\mathrm{p}$ is the part due to slip only, while $\mathbf{F}_*$ is due to lattice stretching and rotation. Denote the unit vector in the slip direction by $\mathbf{s}_\mathrm{o}^\alpha$ and the unit normal to corresponding slip plane in the undeformed configuration by $\mathbf{m}_\mathrm{o}^\alpha$, where α designates the slip system. The vector $\mathbf{s}_\mathrm{o}^\alpha$ is embedded in the lattice, so that it becomes $\mathbf{s}^\alpha = \mathbf{F}_* \cdot \mathbf{s}_\mathrm{o}^\alpha$ in the deformed configuration. The normal to the slip plane in the deformed configuration is defined by the reciprocal vector $\mathbf{m}^\alpha = \mathbf{m}_\mathrm{o}^\alpha \cdot \mathbf{F}_*^{-1}$, i.e.,

$$\mathbf{s}^\alpha = \mathbf{F}_* \cdot \mathbf{s}_\mathrm{o}^\alpha, \quad \mathbf{m}^\alpha = \mathbf{m}_\mathrm{o}^\alpha \cdot \mathbf{F}_*^{-1}\,. \tag{3.28}$$

The velocity gradient in the intermediate configuration is a consequence of the slip rates $\dot\gamma^\alpha$ over n active slip systems, such that

$$\dot{\mathbf{F}}_\mathrm{p} \cdot \mathbf{F}_\mathrm{p}^{-1} = \sum_{\alpha=1}^{n} \dot\gamma^\alpha\, \mathbf{s}_\mathrm{o}^\alpha \otimes \mathbf{m}_\mathrm{o}^\alpha\,. \tag{3.29}$$

Using (3.28), the corresponding tensor in the deformed configuration is

$$\mathbf{F}_* \cdot \left(\dot{\mathbf{F}}_\mathrm{p} \cdot \mathbf{F}_\mathrm{p}^{-1}\right) \cdot \mathbf{F}_*^{-1} = \sum_{\alpha=1}^{n} \left(\mathbf{P}^\alpha + \mathbf{Q}^\alpha\right) \dot\gamma^\alpha\,, \tag{3.30}$$

where the second-order tensors $\mathbf{P}^\alpha$ and $\mathbf{Q}^\alpha$ are defined by (Asaro, 1983)

$$\mathbf{P}^\alpha = (\mathbf{s}^\alpha \otimes \mathbf{m}^\alpha)_{\mathrm{s}}\,, \quad \mathbf{Q}^\alpha = (\mathbf{s}^\alpha \otimes \mathbf{m}^\alpha)_{\mathrm{a}}\,. \tag{3.31}$$

Upon decomposing the lattice velocity gradient $\mathbf{L}_*$ into its symmetric and anti-symmetric parts, the lattice rate of deformation $\mathbf{D}_*$ and the lattice spin $\mathbf{W}_*$, we obtain

$$\mathbf{D} = \mathbf{D}_* + \sum_{\alpha=1}^{n} \mathbf{P}^\alpha \dot{\gamma}^\alpha, \quad \mathbf{W} = \mathbf{W}_* + \sum_{\alpha=1}^{n} \mathbf{Q}^\alpha \dot{\gamma}^\alpha\,. \tag{3.32}$$

Since slip is an isochoric deformation process, the elastic strain energy per unit initial volume can be written as $\Psi_{\mathrm{e}} = \Psi_{\mathrm{e}}(\mathbf{E}_*)$. The function Ψ_{e} is expressed in the coordinate system that has fixed orientation relative to the lattice orientation in $\mathcal{B}_{\mathrm{o}}$ and $\mathcal{B}_{\mathrm{p}}$. It is assumed that elastic properties of the crystal are not affected by crystallographic slip. The symmetric Piola-Kirchhoff stress tensor with respect to the lattice deformation is then

$$\mathbf{T}_* = \frac{\partial \Psi_{\mathrm{e}}}{\partial \mathbf{E}_*}\,. \tag{3.33}$$

The stress tensor $\mathbf{T}_*$ is related to the Kirchhoff stress $\boldsymbol{\tau}$ by

$$\mathbf{T}_* = \mathbf{F}_*^{-1} \cdot \boldsymbol{\tau} \cdot \mathbf{F}_*^{-T}\,. \tag{3.34}$$

The plastic incompressibility is assumed ($\det \mathbf{F}_* = \det \mathbf{F}$). The rate of the Piola-Kirchhoff stress $\dot{\mathbf{T}}_*$ can be expressed in terms of the convected rate of the Kirchhoff stress as

$$\dot{\mathbf{T}}_* = \mathbf{F}_*^{-1} \cdot \overset{\blacktriangle}{\boldsymbol{\tau}} \cdot \mathbf{F}_*^{-T} \quad \overset{\blacktriangle}{\boldsymbol{\tau}} = \dot{\boldsymbol{\tau}} - \mathbf{L}_* \cdot \boldsymbol{\tau} - \boldsymbol{\tau} \cdot \mathbf{L}_*{}^T\,. \tag{3.35}$$

It can be readily verified that

$$\overset{\bullet}{\boldsymbol{\tau}} = \overset{\circ}{\boldsymbol{\tau}} + \sum_{\alpha=1}^{n} (\mathbf{Q}^\alpha \cdot \boldsymbol{\tau} - \boldsymbol{\tau} \cdot \mathbf{Q}^\alpha)\, \dot{\gamma}^\alpha\,. \tag{3.36}$$

On the other hand, taking the time derivative in Eq. (3.33), there follows

$$\dot{\mathbf{T}}_* = \bar{\boldsymbol{\Lambda}}_* : \dot{\mathbf{E}}_*\,, \quad \bar{\boldsymbol{\Lambda}}_* = \frac{\partial^2 \Psi_{\mathrm{e}}}{\partial \mathbf{E}_* \otimes \partial \mathbf{E}_*}\,. \tag{3.37}$$

Substituting the first of (3.35) into Eq. (3.37), we deduce

$$\overset{\blacktriangle}{\boldsymbol{\tau}} = \bar{\boldsymbol{\mathcal{L}}} : \mathbf{D}_* , \quad \bar{\boldsymbol{\mathcal{L}}} = \mathbf{F}_* \mathbf{F}_* \bar{\boldsymbol{\Lambda}}_* \mathbf{F}_*^T \mathbf{F}_*^T . \tag{3.38}$$

If the Jaumann rate corotational with the lattice spin $\mathbf{W}_*$ is used, Eq. (3.38) can be recast in the form

$$\overset{\bullet}{\boldsymbol{\tau}} = \boldsymbol{\mathcal{L}} : \mathbf{D}_* . \tag{3.39}$$

The relationship between the corresponding elastic moduli tensors is specified by an equation such as (3.10). Along elastic branch of the response (elastic unloading from elastoplastic state), the total and lattice velocity gradients coincide, so that $\mathbf{L}_* = \mathbf{L}$ and $\overset{\bullet}{\boldsymbol{\tau}} = \overset{\circ}{\boldsymbol{\tau}}$.

The rate-type constitutive framework for elastoplastic loading of the single crystal is obtained by substituting Eq. (3.36) into Eq. (3.39). The result is

$$\overset{\circ}{\boldsymbol{\tau}} = \boldsymbol{\mathcal{L}} : \mathbf{D} - \sum_{\alpha=1}^{n} \mathbf{C}^{\alpha} \dot{\gamma}^{\alpha} , \tag{3.40}$$

where

$$\mathbf{C}^{\alpha} = \boldsymbol{\mathcal{L}} : \mathbf{P}^{\alpha} + (\mathbf{Q}^{\alpha} \cdot \boldsymbol{\tau} - \boldsymbol{\tau} \cdot \mathbf{Q}^{\alpha}) . \tag{3.41}$$

The elastic part of the stress rate $\overset{\circ}{\boldsymbol{\tau}}$ is defined by

$$(\overset{\circ}{\boldsymbol{\tau}})_{\mathrm{e}} = \boldsymbol{\mathcal{L}} : \mathbf{D} , \tag{3.42}$$

since only the remaining part of the stress rate depends on the slip rates $\dot{\gamma}^{\alpha}$. This is the plastic part

$$(\overset{\circ}{\boldsymbol{\tau}})_{\mathrm{p}} = - \sum_{\alpha=1}^{n} \mathbf{C}^{\alpha} \dot{\gamma}^{\alpha} . \tag{3.43}$$

For rate-independent materials it is commonly assumed that plastic flow occurs on the slip system when the resolved shear stress (Schmid stress) on that system reaches the critical value, i.e., $\tau^{\alpha} = \tau^{\alpha}_{\mathrm{cr}}$, where $\tau^{\alpha} = \mathbf{P}^{\alpha} : \boldsymbol{\tau} = \mathbf{s} \cdot \boldsymbol{\tau} \cdot \mathbf{m}$. The rate of change of the critical value of the

resolved shear stress on a given slip system is defined by the hardening law

$$\dot{\tau}_{\rm cr}^{\alpha} = \sum_{\beta=1}^{n_0} h_{\alpha\beta}\,\dot{\gamma}^{\beta}\,, \quad \alpha = 1, 2, \ldots, N\,, \tag{3.44}$$

where N is the total number of all available slip systems, and n_0 is the number of critical (potentially active) slip systems, for which $\tau^{\alpha} = \tau_{\rm cr}^{\alpha}$. It can be shown that

$$\dot{\gamma}^{\alpha} = \sum_{\beta=1}^{n} g_{\alpha\beta}^{-1}\,\mathbf{C}^{\beta} : \mathbf{D}\,, \tag{3.45}$$

where $n \leq n_0$ is the number of active slip systems, and

$$g_{\alpha\beta} = h_{\alpha\beta} + \mathbf{C}^{\alpha} : \mathbf{P}^{\beta}\,. \tag{3.46}$$

It is assumed that the inverse matrix whose components are designated by $g_{\alpha\beta}^{-1}$ exists. Substituting into Eq. (3.43) and combining with Eq. (3.42) then gives the elastoplastic constitutive equation

$$\overset{\circ}{\boldsymbol{\tau}} = \left(\boldsymbol{\mathcal{L}} - \sum_{\alpha=1}^{n}\sum_{\beta=1}^{n} g_{\alpha\beta}^{-1}\,\mathbf{C}^{\alpha} \otimes \mathbf{C}^{\beta} \right) : \mathbf{D}\,. \tag{3.47}$$

4. BIOMECHANICS

The analysis of the stress-modulated growth of living tissues, bones, and other biomaterials has been an important research topic in biomechanics during past several decades. Early work includes a study of the relationship between the mechanical loads and uniform growth by Hsu (1968), and a study of the mass deposition and resorption processes in a living bone (hard tissue) by Cowin and Hegedus (1976). The latter work provided a set of governing equations of the so-called adaptive elasticity theory, in which an elastic material adopts its structure to applied loading. Fundamental contribution was further made by Skalak *et al.* (1982) in their analytical description of the volumetrically distributed mass growth, and the mass growth by deposition or resorption

on the surface. The origin and the role of residual stresses in biological tissues have been examined both analytically and experimentally by many researchers. The review papers by Humphrey (1995) and Taber (1995) contain an extensive list of references. In contrast to hard tissues, which undergo only small deformations, soft tissues such as blood vessels and tendons can experience large deformations. An important contribution to the general study of finite volumetric growth in soft elastic tissues was made by Rodrigez, Hoger, and McCulloch (1994), who introduced the multiplicative decomposition of the corresponding deformation gradient into its elastic and growth parts. Subsequent work includes the studies by Taber and Eggers (1996), Taber and Perucchio (2000), Chen and Hoger (2000), Klisch, Van Dyke, and Hoger (2001), and Lubarda and Hoger (2001).

The deformation gradient is due to the mass growth and deformation by externally applied and the growth-induced stress. The intermediate configuration is obtained by instantaneous elastic destressing of the current configuration to zero stress (Fig. 3), so that

$$\mathbf{F} = \mathbf{F}_{\mathrm{e}} \cdot \mathbf{F}_{\mathrm{g}} . \tag{4.1}$$

This decomposition, formally analogous to previously considered thermoelastic and elastoplastic decompositions, was first introduced in biomechanics by Rodrigez, Hoger, and McCulloch (1994). The modification of the decomposition to account for the residually stressed reference configuration was introduced by Hoger, Van Dyke, and Lubarda (2001).

If the mass of an infinitesimal volume element in the initial configuration is $\mathrm{d}m^{\circ} = \rho^{\circ}\,\mathrm{d}V^{\circ}$, the mass of the corresponding element in the configurations $\mathcal{B}_g$ and $\mathcal{B}$ is

$$\mathrm{d}m = \rho_{\mathrm{g}}\,\mathrm{d}V_{\mathrm{g}} = \rho\,\mathrm{d}V . \tag{4.2}$$

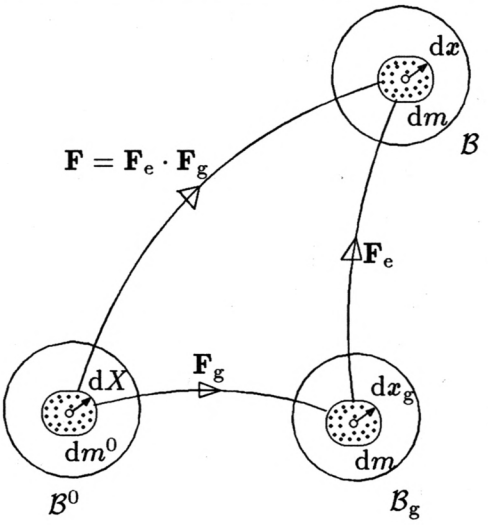

Figure 3: Schematic representation of the multiplicative decomposition of deformation gradient into its elastic and growth parts. The mass of an infinitesimal volume element in $\mathcal{B}^{\circ}$ is $\mathrm{d}m^{\circ}$, while the corresponding mass in $\mathcal{B}_{\mathrm{g}}$ and $\mathcal{B}$ is $\mathrm{d}m$.

Since

$$\mathrm{d}m = \mathrm{d}m^{\circ} + \int_0^t r_{\mathrm{g}}^{\circ}\,\mathrm{d}\tau\,\mathrm{d}V^{\circ}\,, \tag{4.3}$$

where r_{g}^0 is the time rate of mass growth per unit initial volume, and

$$\mathrm{d}V_{\mathrm{g}} = J_{\mathrm{g}}\,\mathrm{d}V^{\circ}\,, \quad J_{\mathrm{g}} = \det\mathbf{F}_{\mathrm{g}}\,, \tag{4.4}$$

we have

$$\rho_{\mathrm{g}}\,J_{\mathrm{g}} = \rho^{\circ} + \int_0^t r_{\mathrm{g}}^{\circ}\,\mathrm{d}\tau\,. \tag{4.5}$$

In addition,

$$\rho_{\mathrm{g}}\,J_{\mathrm{g}} = \rho\,J\,, \quad \rho_{\mathrm{g}} = \rho\,J_{\mathrm{e}}\,, \tag{4.6}$$

because $\mathrm{d}V = J_{\mathrm{e}}\,\mathrm{d}V_{\mathrm{g}}$ and $J = J_{\mathrm{e}}J_{\mathrm{g}}$, where $J_{\mathrm{e}} = \det\mathbf{F}_{\mathrm{e}}$.

Consider an isothermal deformation and growth process. Denote the set of structural tensors that describe the state of elastic anisotropy in both the initial and intermediate configurations by $\mathbf{S}^{\circ}$. For simplicity, we assume that the state of elastic anisotropy remains unaltered during the growth and deformation. The elastic strain energy per unit current mass is then given by an isotropic function of the elastic strain $\mathbf{E}_{\mathrm{e}}$ and the tensors $\mathbf{S}^{\circ}$, i.e., $\psi_{\mathrm{e}} = \psi_{\mathrm{e}}\left(\mathbf{E}_{\mathrm{e}}, \mathbf{S}^{\circ}, \rho_{\mathrm{g}}^{\circ}\right)$. It follows that

$$\boldsymbol{\tau} = \mathbf{F}_{\mathrm{e}} \cdot \frac{\partial(\rho_{\mathrm{g}}^{\circ}\psi_{\mathrm{e}})}{\partial \mathbf{E}_{\mathrm{e}}} \cdot \mathbf{F}_{\mathrm{e}}^{T} = 2\mathbf{F}_{\mathrm{e}} \cdot \frac{\partial(\rho_{\mathrm{g}}^{\circ}\psi_{\mathrm{e}})}{\partial \mathbf{C}_{\mathrm{e}}} \cdot \mathbf{F}_{\mathrm{e}}^{T}. \tag{4.7}$$

4.6 Partition of the Rate of Deformation

The velocity gradient is

$$\mathbf{L} = \dot{\mathbf{F}}_{\mathrm{e}} \cdot \mathbf{F}_{\mathrm{e}}^{-1} + \mathbf{F}_{\mathrm{e}} \cdot \left(\dot{\mathbf{F}}_{\mathrm{g}} \cdot \mathbf{F}_{\mathrm{g}}^{-1}\right) \cdot \mathbf{F}_{\mathrm{e}}^{-1}. \tag{4.8}$$

The symmetric and antisymmetric parts of the second term on the right-hand side will be conveniently denoted by

$$\mathbf{d}_{\mathrm{g}} = \left[\mathbf{F}_{\mathrm{e}} \cdot \left(\dot{\mathbf{F}}_{\mathrm{g}} \cdot \mathbf{F}_{\mathrm{g}}^{-1}\right) \cdot \mathbf{F}_{\mathrm{e}}^{-1}\right]_{\mathrm{s}}. \tag{4.9}$$

$$\boldsymbol{\omega}_{\mathrm{g}} = \left[\mathbf{F}_{\mathrm{e}} \cdot \left(\dot{\mathbf{F}}_{\mathrm{g}} \cdot \mathbf{F}_{\mathrm{g}}^{-1}\right) \cdot \mathbf{F}_{\mathrm{e}}^{-1}\right]_{\mathrm{a}}. \tag{4.10}$$

The elastic part of the rate of deformation tensor is defined by a kinetic relation

$$\mathbf{D}_{\mathrm{e}} = \boldsymbol{\mathcal{L}}_{\mathrm{e}}^{-1} : \overset{\circ}{\boldsymbol{\tau}}, \quad \overset{\circ}{\boldsymbol{\tau}} = \dot{\boldsymbol{\tau}} - \mathbf{W} \cdot \boldsymbol{\tau} + \boldsymbol{\tau} \cdot \mathbf{W}. \tag{4.11}$$

The remaining part of the rate of deformation will be referred to as the growth part, such that

$$\mathbf{D} = \mathbf{D}_{\mathrm{e}} + \mathbf{D}_{\mathrm{g}}. \tag{4.12}$$

To derive an expression for $\mathbf{D}_{\mathrm{g}}$, we differentiate Eq. (4.7) and obtain

$$\begin{aligned} \dot{\boldsymbol{\tau}} \; &= \left(\dot{\mathbf{F}}_{\mathrm{e}} \cdot \mathbf{F}_{\mathrm{e}}^{-1}\right) \cdot \boldsymbol{\tau} + \boldsymbol{\tau} \cdot \left(\dot{\mathbf{F}}_{\mathrm{e}} \cdot \mathbf{F}_{\mathrm{e}}^{-1}\right)^{T} \\ &\quad + \mathbf{F}_{\mathrm{e}} \cdot \left(\boldsymbol{\Lambda}_{\mathrm{e}} : \dot{\mathbf{E}}_{\mathrm{e}}\right) \cdot \mathbf{F}_{\mathrm{e}}^{T} + \frac{\partial \boldsymbol{\tau}}{\partial \rho_{\mathrm{g}}^{\circ}} r_{\mathrm{g}}^{\circ}, \end{aligned} \tag{4.13}$$

where

$$\mathbf{\Lambda}_{\mathrm{e}} = \frac{\partial^2(\rho_{\mathrm{g}}^{\mathrm{o}}\psi_{\mathrm{e}})}{\partial \mathbf{E}_{\mathrm{e}} \otimes \partial \mathbf{E}_{\mathrm{e}}} = 4\,\frac{\partial^2(\rho_{\mathrm{g}}^{\mathrm{o}}\psi_{\mathrm{e}})}{\partial \mathbf{C}_{\mathrm{e}} \otimes \partial \mathbf{C}_{\mathrm{e}}}\,, \tag{4.14}$$

and

$$\frac{\partial \boldsymbol{\tau}}{\partial \rho_{\mathrm{g}}^{\mathrm{o}}} = \mathbf{F}_{\mathrm{e}} \cdot \frac{\partial^2(\rho_{\mathrm{g}}^{\mathrm{o}}\psi_{\mathrm{e}})}{\partial \mathbf{E}_{\mathrm{e}}\, \partial \rho_{\mathrm{g}}^{\mathrm{o}}} \cdot \mathbf{F}_{\mathrm{e}}^{T} = 2\,\mathbf{F}_{\mathrm{e}} \cdot \frac{\partial^2(\rho_{\mathrm{g}}^{\mathrm{o}}\psi_{\mathrm{e}})}{\partial \mathbf{C}_{\mathrm{e}}\, \partial \rho_{\mathrm{g}}^{\mathrm{o}}} \cdot \mathbf{F}_{\mathrm{e}}^{T}\,. \tag{4.15}$$

The structural tensors $\mathbf{S}^{\mathrm{o}}$ remain constant during the differentiation. Equivalently, Eq. (4.13) can be written as

$$\overset{\circ}{\boldsymbol{\tau}} = \boldsymbol{\mathcal{L}}_{\mathrm{e}} : \left(\dot{\mathbf{F}}_{\mathrm{e}} \cdot \mathbf{F}_{\mathrm{e}}^{-1}\right)_{\mathrm{s}} - \boldsymbol{\omega}_{\mathrm{g}} \cdot \boldsymbol{\tau} + \boldsymbol{\tau} \cdot \boldsymbol{\omega}_{\mathrm{g}} + \frac{\partial \boldsymbol{\tau}}{\partial \rho_{\mathrm{g}}^{\mathrm{o}}}\, r_{\mathrm{g}}^{\mathrm{o}}\,. \tag{4.16}$$

The rectangular components of the elastic moduli tensor $\boldsymbol{\mathcal{L}}_{\mathrm{e}}$ are defined by Eq. (3.10). Since

$$\left(\dot{\mathbf{F}}_{\mathrm{e}} \cdot \mathbf{F}_{\mathrm{e}}^{-1}\right)_{\mathrm{s}} = \mathbf{D} - \mathbf{d}_{\mathrm{g}}\,, \tag{4.17}$$

equation (4.16) can be rewritten as

$$\boldsymbol{\mathcal{L}}_{\mathrm{e}}^{-1} : \overset{\circ}{\boldsymbol{\tau}} = \mathbf{D} - \mathbf{d}_{\mathrm{g}} - \boldsymbol{\mathcal{L}}_{\mathrm{e}}^{-1} : \left(\boldsymbol{\omega}_{\mathrm{g}} \cdot \boldsymbol{\tau} - \boldsymbol{\tau} \cdot \boldsymbol{\omega}_{\mathrm{g}} - \frac{\partial \boldsymbol{\tau}}{\partial \rho_{\mathrm{g}}^{\mathrm{o}}}\, r_{\mathrm{g}}^{\mathrm{o}}\right). \tag{4.18}$$

According tom Eq. (4.11), the left-hand side is the elastic part of the rate of deformation tensor, so that the growth part is given by

$$\mathbf{D}_{\mathrm{g}} = \mathbf{d}_{\mathrm{g}} + \boldsymbol{\mathcal{L}}_{\mathrm{e}}^{-1} : \left(\boldsymbol{\omega}_{\mathrm{g}} \cdot \boldsymbol{\tau} - \boldsymbol{\tau} \cdot \boldsymbol{\omega}_{\mathrm{g}} - \frac{\partial \boldsymbol{\tau}}{\partial \rho_{\mathrm{g}}^{\mathrm{o}}}\, r_{\mathrm{g}}^{\mathrm{o}}\right). \tag{4.19}$$

4.7 Isotropic Mass Growth

For isotropic materials, which remain isotropic during the mass growth and deformation, the elastic strain energy is an isotropic function of the elastic deformation tensor $\mathbf{C}_{\mathrm{e}}$, thus the function of its principal invariants,

$$\psi_{\mathrm{e}} = \psi_{\mathrm{e}}\left(\mathbf{C}_{\mathrm{e}},\, \rho_{\mathrm{g}}^{\mathrm{o}}\right) = \psi_{\mathrm{e}}\left(I_C,\, II_C,\, III_C,\, \rho_{\mathrm{g}}^{\mathrm{o}}\right). \tag{4.20}$$

The principal invariants are

$$I_C = \operatorname{tr} \mathbf{C}_e\,, \quad II_C = \frac{1}{2}\Big[\operatorname{tr}\left(\mathbf{C}_e^2\right) - \left(\operatorname{tr} \mathbf{C}_e\right)^2\Big], \quad III_C = \det \mathbf{C}_e\,. \tag{4.21}$$

Consequently, the Kirchhoff stress is from Eq. (4.7)

$$\boldsymbol{\tau} = 2\left(c_2\,\mathbf{I} + c_0\,\mathbf{B}_e + c_1\,\mathbf{B}_e^2\right), \tag{4.22}$$

where $\mathbf{B}_e = \mathbf{F}\cdot\mathbf{F}_e^T$ is the left Cauchy–Green deformation tensor. The scalar coefficients are here

$$\begin{aligned} c_0 &= \frac{\partial(\rho_g^o\psi_e)}{\partial I_C} - I_C\,\frac{\partial(\rho_g^o\psi_e)}{\partial II_C}\,, \\ c_1 &= \frac{\partial(\rho_g^o\psi_e)}{\partial II_C}\,, \quad c_2 = III_C\,\frac{\partial(\rho_g^o\psi_e)}{\partial III_C}\,. \end{aligned} \tag{4.23}$$

If the mass growth occurs isotropically, the growth part of the deformation gradient is

$$\mathbf{F}_g = \vartheta_g\,\mathbf{I}\,, \tag{4.24}$$

where ϑ_g is the isotropic stretch ratio due to volumetric mass growth. It readily follows that the velocity gradient in the intermediate configuration is

$$\dot{\mathbf{F}}_g\cdot\mathbf{F}_g^{-1} = \frac{\dot{\vartheta}_g}{\vartheta_g}\,\mathbf{I}\,. \tag{4.25}$$

The velocity gradient in the configuration $\mathcal{B}$ is accordingly

$$\mathbf{L} = \dot{\mathbf{F}}_e\cdot\mathbf{F}_e^{-1} + \frac{\dot{\vartheta}_g}{\vartheta_g}\,\mathbf{I}\,. \tag{4.26}$$

Since $\boldsymbol{\omega}_g = \mathbf{0}$, the growth part of the rate of deformation tensor becomes

$$\mathbf{D}_g = \frac{\dot{\vartheta}_g}{\vartheta_g}\,\mathbf{I} - \boldsymbol{\mathcal{L}}_e^{-1} : \left(\frac{\partial\boldsymbol{\tau}}{\partial\rho_g^o}\,r_g^o\right), \tag{4.27}$$

which follows from Eq. (4.19). The rectangular components of the elastic moduli tensor $\boldsymbol{\mathcal{L}}_e$ can be found in Lubarda and Hoger (2001).

Various forms of the strain energy function were proposed in the literature for different biological materials. The articles by Holzapfel,

Gasser, and Ogden (2000), and Sacks (2000) contain a number of pertinent references. Following Fung's (1973, 1995) proposal for vascular soft tissues modeled as incompressible elastic materials, the following structure of the elastic strain energy per unit initial volume is adopted

$$\rho_{\mathrm{g}}^{\circ}\psi_{\mathrm{e}} = \frac{1}{2}\alpha_0\Big[\exp(Q) - Q - 1\Big] + \frac{1}{2}q - \frac{1}{2}p\,(III_C - 1)\,. \tag{4.28}$$

Here, Q and q are the polynomials in the invariants of $\mathbf{C}_{\mathrm{e}}$ which include terms up to the fourth order in elastic stretch ratios, i.e.,

$$Q = \alpha_1\,(I_C - 3) + \alpha_2\,(II_C - 3) + \alpha_3\,(I_C - 3)^2\,, \tag{4.29}$$

$$q = \beta_1\,(I_C - 3) + \beta_2\,(II_C - 3) + \beta_3\,(I_C - 3)^2\,. \tag{4.30}$$

The incompressibility constraint is $III_C - 1 = 0$, and the pressure p plays the role of the Lagrangian multiplier. The α's and β's are the material parameters. In order that the intermediate configuration is unstressed, we require that $\beta_1 - 2\,\beta_2 = Jp$.

4.8 Evolution Equation for Stretch Ratio

The constitutive formulation is completed by specifying an appropriate evolution equation for the stretch ratio ϑ_{g}. In the particular, but for the tissue mechanics important special case, when the growth takes place in a density preserving manner ($\rho_{\mathrm{g}} = \rho^{\circ}$), we have from Eq. (4.25)

$$\mathrm{tr}\left(\dot{\mathbf{F}}_{\mathrm{g}}\cdot\mathbf{F}_{\mathrm{g}}^{-1}\right) = 3\,\frac{\dot{\vartheta}_{\mathrm{g}}}{\vartheta_{\mathrm{g}}} = \frac{\mathrm{r}_{\mathrm{g}}}{\rho}\,. \tag{4.31}$$

Thus, recalling that $r_{\mathrm{g}}/\rho = r_{\mathrm{g}}^{\circ}/\rho_{\mathrm{g}}^{\circ}$, the rate of the mass growth $r_{\mathrm{g}}^{\circ} = \mathrm{d}\rho_{\mathrm{g}}^{\circ}/\mathrm{d}t$ can be expressed in terms of the rate of stretch $\dot{\vartheta}_{\mathrm{g}}$ as

$$r_{\mathrm{g}}^{\circ} = 3\,\rho_{\mathrm{g}}^{\circ}\,\frac{\dot{\vartheta}_{\mathrm{g}}}{\vartheta_{\mathrm{g}}}\,. \tag{4.32}$$

Upon integration of Eq. (4.32), using the initial conditions $\vartheta_{\mathrm{g}}^{\mathrm{o}} = 1$ and $\rho_{\mathrm{g}}^{\mathrm{o}} = \rho^{\mathrm{o}}$, we obtain

$$\rho_{\mathrm{g}}^{\mathrm{o}} = \rho^{\mathrm{o}}\, \vartheta_{\mathrm{g}}^{3} . \tag{4.33}$$

We propose as an evolution equation for the stretch ratio ϑ_{g} the following expression

$$\dot{\vartheta}_{\mathrm{g}} = f_{\vartheta}\left(\vartheta_{\mathrm{g}},\, \mathrm{tr}\, \mathbf{T}_{\mathrm{e}}\right) . \tag{4.34}$$

The tensor $\mathbf{T}_{\mathrm{e}}$ is the symmetric Piola–Kirchhoff stress with respect to intermediate configuration $\mathcal{B}_{\mathrm{g}}$ where the stretch ratio ϑ_{g} is defined. For isotropic mass growth, only spherical part of this tensor is assumed to affect the change of the stretch ratio. This can be expressed in terms of the Cauchy stress $\boldsymbol{\sigma}$ and the elastic deformation as

$$\mathrm{tr}\, \mathbf{T}_{\mathrm{e}} = J_{\mathrm{e}}\, \mathbf{B}_{\mathrm{e}}^{-1} : \boldsymbol{\sigma} . \tag{4.35}$$

The simplest evolution of the stretch ratio incorporates a linear dependence on the stress, such that

$$\dot{\vartheta}_{\mathrm{g}} = k_{\vartheta}\left(\vartheta_{\mathrm{g}}\right) \mathrm{tr}\, \mathbf{T}_{\mathrm{e}} . \tag{4.36}$$

This implies that the growth-equilibrium stress is equal to zero (i.e., $\dot{\vartheta}_{\mathrm{g}} = 0$ when $\mathrm{tr}\, \mathbf{T}_{\mathrm{e}} = 0$). The coefficient k_{ϑ} may be constant, or dependent on ϑ_{g}. For example, k_{ϑ} may take one value during the development of the tissue, and another value during the normal maturity. Yet another value may be characteristic for abnormal conditions, such as occur in thickening of blood vessels under hypertension. Other evolution equations were also suggested in the literature, motivated by the possibilities of growth and resorption. The most well-known is the evolution equation for the mass growth in terms of a nonlinear function of stress, which includes three growth-equilibrium states of stress (Fung, 1990). The material parameters in these expressions should be specified in accordance with experimental data obtained for the particular tissue. This is clearly an essential aspect of the future research. Appealing tests include those with a transmural radial cut through the blood vessel,

which relieves the residual stresses due to differential growth of its inner and outer layers. The opening angle then provides a convenient measure of the circumferential residual strain, as discussed by Liu and Fung (1988, 1989), Humphrey (1995), Taber and Eggers (1996), and others.

5. CONCLUSIONS

The application of the multiplicative decomposition of the deformation gradient in non-linear continuum mechanics is reviewed. The essential features of the resulting constitutive formulations in thermoelasticity, elastoplasticity, and biomechanics are given. In thermoelasticity and phenomenological polycrystalline plasticity the use of the decomposition may be considered as optional, since the respective constitutive formulations can proceed with or without it (e.g., Truesdell and Noll, 1965), although some results derived on the basis of thermoelastic decomposition appear to be more suitable for direct incorporation of experimental data for temperature dependent elastic moduli, thermal expansion, and specific heats (Vujošević and Lubarda, 2002). Also, various kinematic and kinetic aspects in the analysis of the partition of the stress and strain rates in phenomenological elastoplasticity are more transparent when addressed in the framework of the decomposition. This is particularly the case when large elastic deformations accompany plastic deformations, as occurs under explosive and dynamic loadings (Clifton, 1983). Furthermore, there was an important application of the decomposition in damage-elastoplasticity (Lubarda, 1994; Lubarda and Krajcinovic, 1995), where plastic deformation significantly affects the initial elastic properties of the material. In single crystal plasticity the multiplicative decomposition was commonly adopted as the most sound basis for the constitutive analysis of slip-induced large elastoplastic deformations (Havner, 1992; Lubarda, 2002). The application of the decomposition in the analysis of the stress-modulated growth of pseudo-elastic soft tissues in biomechanics, such as blood vessels and tendons, is more recent and least explored (Rodrigez *et al.*, 1995). The extent of the decomposition's utility for such problems, in

spite of some early promising results (Klisch *et al.*, 2001; Lubarda and Hoger, 2001), remains to be seen.

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