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KOVARIJACIJA PROCESA GENERISANOG STATISTIKOM SA VREMENSKIM POMJERANJEM

I z v o d

U radu je nađena asimptotska jednakost za kovarijaciju procesa $\bar{f}_N(\lambda)$, $\lambda \in R$, generisanog statistikom sa vremenskim pomjeranjem. Nađena je odgovarajuća asimptotika i za proces generisan Kolmogorovljevom statistikom.

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ON COVARIANCE OF PROCESS WHICH IS DETERMINED WITH TIME SHIFT STATISTIC

A b s t r a c t

In this paper we establish the asymptotic behaviour of a covariance of the process $\bar{f}_N(\lambda)$, $\lambda \in R$, the process is determined with the time

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shift statistic. The covariance of the process which is determined with Kolmogorov statistic has beeen performed as a consequence of this general approach.

1. UVOD

Statistike tipa Grenander-Rozenblata dugo su bile neprikosnovene u neparametarskim metodama ocjenjivanja spektralne gustine. Optimalnu statistiku na ovoj klasi, pri prirodnim i veoma opštim uslovima, našao je I. Žurbenko [1]. U statistici $\hat{f}_N(\lambda)$ (uobičajena oznaka za statistiku tipa Grenander-Rozenblata) uticaj okolnih frekvencija opada sa asimptotikom reda $\frac{C}{N^2}$, N je dužina serije (dakle, uticaj je velik), pa se na ocjenu reflektuju neregularnosti spektralne gustine. Ovaj nedostatak statistiku $\hat{f}_N(\lambda)$, u nekim situacijama, čini nepodesnom za primjenu. Uz to, često prisutni trendovi i nestacionarni šumovi, ometaju statistiku $\hat{f}_N(\lambda)$ u ocjenjivanju.

Uticaj okolnih frekvencija se znatno smanjuje nakon procedure dubokog glačanja na krajevima uzoračkog niza. Duboko glačanje se ostvaruje primjenom vremenskog prozora. Nestacionarne smetnje se bitno neutrališu osrednjavanjem modifikovanih periodograma.

Kombinujući obje procedure, A. Kolmogorov [1] je konstruisao statistiku koja je prihvaćena pod imenom statistika sa vremenskim pomjeranjem. U ovu statistiku, Kolmogorov je ugradio polinomijalni prozor (tako dobijena statistika je u literaturi poznata kao Kolmogorovljeva statistika). U Kolmogorovljevoj statistici, zavisnost od susjednih frekvencija se svodi na $N^{-\frac{2K}{1+\alpha}}$, gdje K može biti izabrano po želji velikim, a α je Helderov koeficijent spektralne gustine $f(\lambda)$. Statistika sa vremenskim pomjeranjem daje mogućnost provjere stacionarnosti na zadatom pojasu frekvencija. Broj računskih operacija potrebnih za dobijanje ocjene se redukuje na $n^{1+\epsilon} \ln N$, ϵ je proizvoljno malo i ovo je značajno poboljšanje u odnosu na $N^{\frac{1+4\alpha}{1+2\alpha}} \ln N$ operacija potrebnih za dobijanje Grenander-Rozenblatove ocjene.

Stvoreni model oponaša fizičku shemu neprekidno djejstvujuće uskopojasne filtracije realnog signala. Zbog visoke efektivnosti i otpornosti na smetnje, statistika sa vremenskim pomjeranjem je dobro primljena u geofizici, akustici, neurofiziologiji.

U sljedećem odjeljku daćemo matematički formalizovan postupak zadavanja statistike sa vremenskim pomjeranjem. Ovu statistiku, označavaćemo je sa $\bar{f}_N(\lambda)$, je detaljno izučio I. Žurbenko [1].

U našem radu je nađena asimptotika kovarijacije procesa $\bar{f}_N(\lambda)$, $\lambda \in R$. Dobijeni rezultat je neophodan korak na putu ustanovljivanja asimptotskog ponašanja procesa $\bar{f}_N(\lambda)$, $\lambda \in R$.

Kovarijacionu strukturu i svojstva procesa $\hat{f}_N(\lambda)$, $\lambda \in R$, je izučio P. Mladenović [2, 3, 5].

2. OSNOVNI POJMOVI, OZNAKE KOJE SE KORISTE U TEKSTU, REZULTATI

Neka je $X(t)$, $t \in Z$, centrirani stacionarni slučajni niz. Semiinvarijantu niza ćemo označavati sa $S_n(t_1, t_2, \ldots, t_n)$, a semiinvarijantnu spektralnu gustinu sa $f(\lambda_1, \lambda_2, \ldots, \lambda_n)$.

Neka je $a_M(t)$, $t \in Z$, nenegativna funkcija koja je jednaka nuli van segmenta $[0, M]$. Na uzorku $X(Q), X(Q+1), \ldots, X(Q+M)$, definišimo statistiku

$$W_M^Q(\lambda) = \sum_{t=-\infty}^{+\infty} a_M(t-Q)e^{it\lambda}X(t).$$

Označimo sa

$$\varphi_M^Q(x) = \sum_{t=-\infty}^{+\infty} a_M(t-Q)e^{itx}. \tag{2.1}$$

Primijetimo da je $\varphi_M^Q(x) = \varphi_M(x)e^{iQx}$. Ubuduće ćemo $\varphi_M^0(x)$ označavati sa $\varphi_M(x)$.

Koeficijente $a_M(t)$ biramo tako da je $\int_{-\pi}^{\pi} |\varphi_M(x)|^2 dx = 1$, odakle slijedi da je $\sum_{t=0}^{M} a_M^2(t) = \frac{1}{2\pi}$. Koeficijenti $a_M(t)$ obrazuju vremenski prozor.

Statistiku sa vremenskim pomjeranjem $\bar{f}_N(\lambda)$, kojom ocjenjujemo spektralnu gustinu $f(\lambda)$ stacionarnog slučajnog niza $X(t)$, $t \in Z$, na uzorku $X(0), X(1), \ldots, X(N)$, definišemo sa

$$\bar{f}_N(\lambda) = \frac{1}{T} \sum_{k=0}^{T-1} |W_M^{Lk}(\lambda)|^2,$$

gdje je $N = (T-1)L + M + 1, L, M$ i T su cjelobrojne nenegativne funkcije argumenta N za koje važi $L \ll M \ll N$ ($\ll$ je oznaka za asimptotski "bitno manje") i $LT \sim N$.

Polinomijalni vremenski prozor zadaje se koeficijentima $a_M(t)$, za koje važi

$$a_M(t) = a_{K,P}(t) = \mu(K,P)\Big(\frac{K(P^2-1)}{12\pi}\Big)^{\frac{1}{4}} P^{-K} C_{K,P}(t), \qquad (2.2)$$

gdje je $M = K(P-1)$, P je fiksiran prirodan broj i koeficijenti $C_{K,P}(t)$ se zadaju sa

$$\sum_{t=0}^{K(P-1)} z^t C_{K,P}(t) = (1 + z + \ldots + z^{P-1})^K = \Big(\frac{1-z^P}{1-z}\Big)^K.$$

Iz (2.1) i (2.2) slijedi da je

$$\varphi_M(x) = \varphi_{K,P}(x) = \mu(K,P)\Big(\frac{K(P^2-1)}{12\pi}\Big)^{\frac{1}{4}} \Big(\frac{1-e^{iPx}}{P(1-e^{ix})}\Big)^K,$$

te je polinomijalno jezgro

$$|\varphi_{K,P}(x)|^2 = \mu^2(K,P)\Big(\frac{K(P^2-1)}{12\pi}\Big)^{\frac{1}{2}} \Big(\frac{\sin^2\frac{Px}{2}}{P^2\sin^2\frac{x}{2}}\Big)^K.$$

Polinomijalno jezgro ima blago zaobljen maksimum u nuli i veoma je uglačano na krajevima intervala $(-\pi, \pi)$.

Koristićemo oznaku $\eta(\lambda)$, gdje je $\eta(\lambda) = \begin{cases} 1, & \lambda = 0 \ (\mathrm{mod}\pi) \\ 0, & \lambda \neq 0 \ (\mathrm{mod}\pi) \end{cases}.$

Definicija 2.1. Označimo sa $\mathcal{F}^*$ klasu spektralnih jezgara $|\varphi_M(x)|^2 = \varphi_M(x)\bar{\varphi}_M(x) = \varphi_M(x)\varphi_M(-x)$ (funkcija $\varphi_M(x)$ zadaje se sa (2.2)) za koje važi:

$$1^\circ \qquad \max_x |\varphi_M(x)|^2 \sim W_1 \int_{-\pi}^{\pi} |\varphi_M(x)|^4 dx \sim W_2 M^{\frac{1}{2}},\ N \to \infty,$$

$$M \ll N^{\frac{2}{5}},\ L = o(\max_x |\varphi_M(x)|^2),\ N \to \infty,$$

gdje su W_1 i W_2 pozitivni brojevi koji se mijenjaju pri prolasku jezgra $|\varphi_M(x)|^2$ kroz klasu $\mathcal{F}^*$.

2° Postoji niz $A(N) = A$, čiji su članovi prirodni brojevi, takav da je $\frac{LA}{N} = o(1), A \to \infty$ i

$$\sum_{t=0}^{M} a_M(t) \ll \sqrt{\frac{N}{AM}},$$

3° $$\sum_{1 \leq k \leq 2L} \sup_{\frac{k\pi}{2L} \leq x \leq \pi} |\varphi_M(x)||\varphi_M(x+\psi)| = O(L),$$

i ova asimptotska jednakost važi ravnomjerno po ψ.

Definicija 2.2. Za spektralno jezgro $|\varphi_M(x)|^2$ kažemo da pripada klasi $\mathcal{F}^{**}$ ako pripada klasi $\mathcal{F}^*$ i uz to se može ravnomjerno na $[-\pi, \pi]$ aproksimirati jezgrom oblika $G_M(x) = K_M A_M \Phi(A_M x)$, gdje je $K_M = \left(\int_{-A_M\pi}^{A_M\pi} \Phi(x)dx\right)^{-1}$, a funkcije $\Phi(x)$ koje zadaju jezgro $G_M(x)$ su nenegativne, neprekidne, parne, $\int_{-\infty}^{\infty} \Phi(x)dx = 1$, dok je $\int_{-\pi}^{\pi} |\varphi_M(x)|^4 \sim W_3 A_M$ i W_3 je pozitivan broj koji se mijenja pri prolasku jezgra $|\varphi_M(x)|^2$ kroz klasu $\mathcal{F}^{**}$.

Spektralna jezgra $|\varphi_M(x)|^2$ i $G_M(x)$ su parna, zadata su na $[-\pi, \pi]$ i mogu se produžiti u periodične funkcije (periode 2π) koje su zadate na realnoj pravoj. Sada se uslov ravnomjerne aproksimacije na $[-\pi, \pi]$ (iz Definicije 2.2) može uopštiti do uslova ravnomjerne aproksimacije na R.

Mi ćemo raditi sa neprekidnom spektralnom gustinom $f(\lambda)$, $-\pi \leq \lambda \leq \pi$, koja se može neprekidno i periodično (sa periodom 2π) produžiti u funkciju čiji argument prolazi skupom realnih brojeva. Proces $\bar{f}_N(\lambda)$ koji je zadat na parametarskom skupu $\Lambda = [-\pi, \pi]$, može se prirodno proširiti u proces $\bar{f}_N(\lambda)$ čiji parametar λ prolazi skupom R. Proces $\bar{f}_N(\lambda)$, $\lambda \in R$, je periodičan i njegova perioda je 2π.

Prije nego što formulišemo našu centralnu teoremu, evidentirajmo dvije pomoćne leme.

Lema 2.1. Ako postoji niz prirodnih brojeva $A(N) = A$ za koji važi $A \to \infty, \frac{A}{N} \to 0, N \to \infty$ i $\sum_{t=0}^{M} a_M(t) \ll \sqrt{\frac{N}{MA}}$, tada važi asimptotska jednakost

$$|\int_{-\pi}^{\pi} [\varphi_M(x)\varphi_M(-x+\psi)\varphi_M(u-x)\varphi_M(-u+x-\psi)-$$
$$-\varphi_M(x)\varphi_M(-x) \times \varphi_M(x-\psi)\varphi_M(-x+\psi)]dx| = o(\max_x |\varphi_M(x)|^2),$$

kada $N \to \infty$, ravnomjerno po ψ za $|u| \leq \frac{A}{N}$.

Lema 2.2. U uslovima koji važe u Lemi 2.1 važi

$$
\begin{aligned}
&\Big|\int_{-\pi}^{\pi}[|\varphi_M(x)||\varphi_M(-x+\psi)||\varphi_M(u-x)||\varphi_M(-u+x-\psi)|-\\
&-|\varphi_M(x)||\varphi_M(-x)|\times|\varphi_M(x-\psi)||\varphi_M(-x+\psi)|]dx\Big| = o(\max_x|\varphi_M(x)|^2),
\end{aligned}
$$

kada $N \to \infty$, ravnomjerno po ψ za $|u| \leq \frac{A}{N}$.

Teorema 2.1. Neka za spektralne gustine stacionarnog slučajnog niza $X(t)$, $t \in Z$, važi $\sup_x|f'(x)| = C_1 < \infty$, $\sup_{x_1,x_2,x_3,x_4}|f(x_1,x_2,x_3,x_4)| = C_2 < \infty$ i neka spektralno jezgro $|\varphi_M(x)|^2$ pripada klasi $\mathcal{F}^*$. Tada važi asimptotska jednakost

$$
\begin{aligned}
cov(\bar{f}_N(\lambda_1), \bar{f}_N(\lambda_2)) &= \frac{2\pi}{N}f(\lambda_1)f(\lambda_2)\int_{-\pi}^{\pi}|\varphi_M(x)|^2(|\varphi_M(x+\lambda_1-\lambda_2)|^2+\\
&+|\varphi_M(x+\lambda_1+\lambda_2)|^2)dx+|\lambda_1-\lambda_2|^*O\Big(\frac{1}{N}\int_{-\pi}^{\pi}|\varphi_M(x)|^2|\varphi_M(x+\lambda_1-\lambda_2)|^2dx\Big)+\\
&+|\lambda_1+\lambda_2|^*O\Big(\frac{1}{N}\int_{-\pi}^{\pi}|\varphi_M(x)|^2|\varphi_M(x+\lambda_1+\lambda_2)|^2dx\Big)+o\left(\frac{1}{N}\int_{-\pi}^{\pi}|\varphi_M(x)|^4dx\right),
\end{aligned}
$$

kada $N \to \infty$, ravnomjerno po λ_1 i λ_2. Funkcija $|x|^*$ zadaje se sa

$$
|x|^* = \begin{cases} \dots & \\ |x+2\pi|, & -3\pi < x < -\pi \\ |x|, & -\pi < x < \pi \\ |x-2\pi|, & \pi < x < 3\pi \\ \dots & \end{cases}
$$

Oslanjajući se na Teoremu 2.1, možemo ustanoviti sljedeće dvije teoreme.

Teorema 2.2. Ako spektralno jezgro $|\varphi_M(x)|^2$ pripada klasi $\mathcal{F}^{**}$, ako je $G_M(x) = K_M A_M \Phi(A_M x)$ spektralno jezgro koje ga ravnomjerno aproksimira i ako za ocjenjivanu spektralnu gustinu važe uslovi iz Teoreme 2.1, tada važi asimptotska jednakost

$$cov(\bar{f}_N(\lambda_1), \bar{f}_N(\lambda_2)) = \frac{2\pi}{N} f(\lambda_1) f(\lambda_2) \int_{-\pi}^{\pi} G_M(x)(G_M(x+\lambda_1-\lambda_2)+$$

$$+G_M(x+\lambda_1+\lambda_2))dx + |\lambda_1-\lambda_2|^* O\Big(\frac{1}{N}\int_{-\pi}^{\pi} G_M(x)G_M(x+\lambda_1-\lambda_2)dx\Big)+$$

$$+|\lambda_1+\lambda_2|^* O\Big(\frac{1}{N}\int_{-\pi}^{\pi} G_M(x)G_M(x+\lambda_1+\lambda_2)dx\Big) + o\Big(\frac{1}{N}\int_{-\pi}^{\pi} G_M^2(x)dx\Big),$$

kada $N \to \infty$, ravnomjerno po λ_1 i λ_2.

Teorema 2.3. Neka važe uslovi iz Teoreme 2.2. Tada je

a) $$\lim_{N\to\infty} \frac{N}{A_M} cov\Big(\bar{f}_N(\lambda), \bar{f}_N(\lambda - \frac{\alpha}{A_M})\Big) =$$
$$= 2\pi f^2(\lambda) \int_{-\infty}^{\infty} \Phi(t)\Phi(t+\alpha)dt \cdot (1+\eta(\lambda)).$$

b) $$\lim_{N\to\infty} \frac{N}{A_M} cov\Big(\bar{f}_N\Big(\frac{\lambda_1}{A_M}\Big), \bar{f}_N\Big(\frac{\lambda_2}{A_M}\Big)\Big) =$$
$$= 2\pi f^2(0) \int_{-\infty}^{\infty} [\Phi(t-\lambda_1)\Phi(t+\lambda_2) + \Phi(t+\lambda_1)\Phi(t+\lambda_2)]dt,$$
gdje je $-A_M\pi \le \lambda_1, \lambda_2 \le A_M\pi$.

Kako su dokazi upravo formulisanih teorema veoma slični, biće demonstriran dokaz samo Teoreme 2.2.

Jednostavnom analizom se pokazuje da polinomijalno jezgro $|\varphi_{K,P}(x)|^2$ pripada klasi $\mathcal{F}^{**}$ i da se ravnomjerno na $[-\pi, \pi]$ može aproksimirati jezgrom

$$G_M(x) = K_M\sqrt{K(P^2-1)}\Phi\Big(\sqrt{K(P^2-1)}x\Big),$$

gdje je $\Phi(t) = \frac{1}{\sqrt{12\pi}} e^{-\frac{t^2}{12}}$ i $K_M = \int_{-\sqrt{K(P^2-1)}}^{\sqrt{K(P^2-1)}} \Phi(t)dt$.

Sada, kao neposrednu posljedicu Teoreme 2.3, možemo formulisati sljedeću teoremu.

Teorema 2.4. Ako za ocjenjivanu spektralnu gustinu važe uslovi iz Teoreme 2.1 i ako je $\bar{f}_N(\lambda)$ Kolmogorovljeva statistika, tada je

a) $$\lim_{N\to\infty} \frac{N}{\sqrt{K(P^2-1)}} cov\left(\bar{f}_N(\lambda), \bar{f}_N\left(\lambda - \frac{\alpha}{\sqrt{K(P^2-1)}}\right)\right) =$$
$$= \frac{f^2(\lambda)}{6} \int_{-\infty}^{\infty} e^{-\frac{t^2}{12}} e^{-\frac{(t-\alpha)^2}{12}} dt(1+\eta(\lambda)).$$

b) $$\lim_{N\to\infty} \frac{N}{\sqrt{K(P^2-1)}} cov\left(\bar{f}_N\left(\frac{\lambda_1}{\sqrt{K(P^2-1)}}\right), \bar{f}_N\left(\frac{\lambda_1}{\sqrt{K(P^2-1)}}\right)\right) =$$
$$= \frac{f^2(0)}{6} \int_{-\infty}^{\infty} e^{-\frac{(t-\lambda_2)^2}{12}} \cdot \left[e^{-\frac{(t-\lambda_1)^2}{12}} + e^{-\frac{(t+\lambda_1)^2}{12}}\right] dt,$$
gdje je $-\sqrt{K(P^2-1)}\pi \le \lambda_1, \lambda_2 \le \sqrt{K(P^2-1)}\pi$.

3. DOKAZI

Dokaz Leme 2.1. Neka je $g_M(x) = \varphi_M(-x)\varphi_M(x-\psi)$. Sada je $g_M(x-u) = \varphi_M(u-x)\varphi_M(x-u-\psi)$, te je

$$\left|\int_{-\pi}^{\pi} [\varphi_M(x)\varphi_M(-x+\psi)\varphi_M(u-x)\varphi_M(-u+x-\psi) - \right.$$
$$\left. -\varphi_M(x)\varphi_M(-x)\varphi_M(x-\psi)\varphi_M(-x+\psi)]dx\right| =$$
$$= \left|\int_{-\pi}^{\pi} \varphi_M(x)\varphi_M(-x+\psi)[\varphi_M(u-x)\varphi_M(-u+x-\psi) - \varphi_M(-x)\varphi_M(x-\psi)]dx\right| \le$$
$$\le \max_x |\varphi_M(x)|^2 \int_{-\pi}^{\pi} |g_M(x) - g_M(x-u)|dx \le \max_x |\varphi_M(x)|^2 \frac{A}{N} \int_{-\pi}^{\pi} |g'_M(x)|dx.$$

Međutim,

$$\begin{aligned} g_M(x) &= \sum_{0\le t\le M} a_M(t)e^{-itx} \sum_{0\le s\le M} a_M(s)e^{is(x-\psi)} = \\ &= \sum_{0\le s,t\le M} a_M(t)a_M(s)e^{ix(s-t)-is\psi}, \end{aligned}$$

te je

$$|g'_M(x)| \le \sum_{s,t=0}^{M} a_M(t)a_M(s)|s-t| \le M\Big(\sum_{t=0}^{M} a_M(t)\Big)^2 \ll M\frac{N}{MA} = \frac{N}{A}.$$

Dakle,

$$\begin{aligned} &|\int_{-\pi}^{\pi}[\varphi_M(x)\varphi_M(-x+\psi)\varphi_M(u-x)\varphi_M(-u+x-\psi) - \varphi_M(x)\varphi_M(-x)\times \\ &\times\varphi_M(x-\psi)\varphi_M(-x+\psi)]dx| \ll \max_x|\varphi_M(x)|^2\tfrac{N}{A}\tfrac{A}{N} = \max_x|\varphi_M(x)|^2. \end{aligned}$$

Ovim je lema dokazana.

Dokaz Leme 2.2. Dokaz je identičan dokazu Leme 2.1, osim dijela u kojem se pojavljuje

$$\begin{aligned} &||\varphi_M(u-x)||\varphi_M(-u+x-\psi)| - |\varphi_M(-x)||\varphi_M(x-\psi)|| = \\ &= ||g(x)| - |g(x-u)|| \le |g(x) - g(x-u)|. \end{aligned}$$

Nakon ove ocjene, dokaz teče analogno dokazu Leme 2.1.

Dokaz Teoreme 2.1. Primjenom uobičajene tehnike imamo

$$cov(\bar{f}_N(\lambda_1), \bar{f}_N(\lambda_2)) = \frac{1}{T^2}\sum_{k_1,k_2=0}^{T-1} cov(|W_M^{Lk_1}(\lambda_1)|^2, |W_M^{Lk_2}(\lambda_2)|^2) =$$

$$= \frac{1}{T^2}\sum_{\substack{k_1=0\\k_2=0}}^{T-1}\sum_{\substack{t_1,t_2=-\infty\\t_3,t_4=-\infty}}^{+\infty} a_M(t_1-Lk_1)a_M(t_2-Lk_1)a_M(t_3-Lk_2)a_M(t_4-Lk_2)\times$$

$$\times e^{i[(t_1-t_2)\lambda_1+(t_4-t_3)\lambda_2]}[EX(t_1)X(t_2)X(t_3)X(t_4)-$$

$$-EX(t_1)X(t_2)EX(t_3)X(t_4)] =$$

$$=\frac{1}{T^2}\sum_{\substack{k_1=0\\k_2=0}}^{T-1}\sum_{\substack{t_1,t_2=-\infty\\t_3,t_4=-\infty}} a_M(t_1-Lk_1)a_M(t_2-Lk_1)a_M(t_3-Lk_2)a_M(t_4-Lk_2)\times$$

$$\times e^{i[(t_1-t_2)\lambda_1+(t_4-t_3)\lambda_2]}[S_4(t_1,t_2,t_3,t_4)+S_2(t_1,t_3)S_2(t_2,t_4)+$$

$$+S_2(t_1,t_4)S_2(t_2,t_3)]=$$

$$=\frac{1}{T^2}\sum_{k_1,k_2=0}^{T-1}\left\{\int\limits_{\Pi^4}\varphi_M^{Lk_1}(x_1+\lambda_1)\varphi_M^{Lk_1}(x_2-\lambda_1)\varphi_M^{Lk_2}(x_3-\lambda_2)\varphi_M^{Lk_2}(x_4+\lambda_2)\times\right.$$

$$\times\delta^*(x_1+x_2+x_3+x_4)f(x_1,x_2,x_3,x_4)dx_1dx_2dx_3dx_4+\int\limits_{-\pi}^{\pi}f(x)\varphi_M^{Lk_1}(x+\lambda_1)\times$$

$$\times\varphi_M^{Lk_2}(-x-\lambda_2)dx\int\limits_{-\pi}^{\pi}f(y)\varphi_M^{Lk_1}(y-\lambda_1)\varphi_M^{Lk_2}(-y+\lambda_2)dy+$$

$$\left.+\int\limits_{-\pi}^{\pi}f(x)\varphi_M^{Lk_1}(x+\lambda_1)\varphi_M^{Lk_2}(\lambda_2-x)dx\int\limits_{-\pi}^{\pi}\varphi_M^{Lk_1}(y-\lambda_1)\varphi_M^{Lk_2}(-y-\lambda_2)f(y)dy\right\}=$$

$$=\frac{1}{T^2}\int\limits_{\Pi^4}\delta^*(x_1+x_2+x_3+x_4)f(x_1,x_2,x_3,x_4)\varphi_M(x_1+\lambda_1)\varphi_M(x_2-\lambda_1)\varphi_M(x_3-\lambda_2)\times$$

$$\times\varphi_M(x_4+\lambda_2)\frac{\sin\frac{TL(x_1+x_2)}{2}}{\sin\frac{L(x_1+x_2)}{2}}\frac{\sin\frac{TL(x_3+x_4)}{2}}{\sin\frac{L(x_3+x_4)}{2}}e^{\frac{iL(T-1)}{2}(x_1+x_2+x_3+x_4)}dx_1\dots dx_4+$$

$$+\frac{1}{T^2}\int\limits_{-\pi}^{\pi}\int\limits_{-\pi}^{\pi}f(x)f(y)\varphi_M(x+\lambda_1)\varphi_M(-x-\lambda_2)\varphi_M(y-\lambda_1)\cdot$$

$$\cdot\varphi_M(-y+\lambda_2)\frac{\sin^2\frac{TL(x+y)}{2}}{\sin^2\frac{L(x+y)}{2}}dxdy+$$

$$+\frac{1}{T^2}\int\limits_{-\pi}^{\pi}\int\limits_{-\pi}^{\pi}f(x)f(y)\varphi_M(x+\lambda_1)\varphi_M(-x+\lambda_2)\varphi_M(y-\lambda_1)\cdot$$

$$\cdot\varphi_M(-y-\lambda_2)\frac{\sin^2\frac{TL(x+y)}{2}}{\sin^2\frac{L(x+y)}{2}}dxdy,$$

gdje je Π oznaka za interval $(-\pi, \pi)$.

Integrale u posljednjoj sumi označimo redom sa I_1, I_2 i I_3. Ocijenimo integral I_1. Uvodjenjem smjena $x_1 = x_1, x_3 = x_3, x_1 + x_2 = u$ i uz korišćenje očiglednih jednakosti

$$\frac{1}{2\pi N} \sum_{s,t=1}^{N} e^{ix(t-s)} = \frac{1}{2\pi N} \frac{\sin^2 \frac{Nx}{2}}{\sin^2 \frac{x}{2}},$$

$$\int_{-\pi}^{\pi} \frac{1}{2\pi N} \frac{\sin^2 \frac{Nx}{2}}{\sin^2 \frac{x}{2}} dx = \int_{-\pi}^{\pi} \frac{1}{2\pi N} \sum_{s,t=1}^{N} e^{ix(t-s)} dx = 1$$

i nejednakosti Koši-Švarca dobijamo

$$|I_1| \leq \frac{C_2}{T^2} \int_{-2\pi}^{2\pi} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \frac{\sin^2 \frac{TLu}{2}}{\sin^2 \frac{Lu}{2}} |\varphi_M(x_1+\lambda_1)\varphi_M(u-x_1-\lambda_1)\varphi_M(x_3-\lambda_1)\times$$

$$\varphi_M(u+x_3-\lambda_2)|dudx_1dx_3 \leq$$

$$\leq \frac{2\pi C_2}{T} \int_{-2\pi}^{2\pi} \frac{\sin^2 \frac{TLu}{2}}{2\pi T \sin^2 \frac{Lu}{2}} du \int_{-\pi}^{\pi} |\varphi_M(x_1+\lambda_1)\varphi_M(\lambda_1+x_1-u)|dx_1\times$$

$$\times \int_{-\pi}^{\pi} |\varphi(x_3-\lambda_1)\varphi_M(u+x_3-\lambda_2)|dx_3 \leq \frac{4\pi C_2}{T} =$$

$$= o\left(\frac{1}{N}\int_{-\pi}^{\pi} |\varphi_M(x)|^4 dx\right), \tag{3.3}$$

ravnomjerno po λ_1 i λ_2.

Izučimo asimptotsko ponašanje integrala I_2.

$$I_2 = \frac{1}{T^2} \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(x)f(y)\varphi_M(x+\lambda_1)\varphi_M(-x-\lambda_2)\varphi_M(y-\lambda_1)\cdot$$

$$\cdot\varphi_M(-y+\lambda_2)\frac{\sin^2 \frac{TL(x+y)}{2}}{\sin^2 \frac{L(x+y)}{2}} dxdy =$$

$$= \frac{1}{T^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f(x-\lambda_1) f(y+\lambda_1) \varphi_M(-x+\lambda_1-\lambda_2) \varphi_M(y) \cdot$$

$$\cdot \varphi_M(-y-\lambda_1+\lambda_2) \frac{\sin^2 \frac{TL(x+y)}{2}}{\sin^2 \frac{L(x+y)}{2}} dx dy.$$

Uvedimo promjenljive x i u, $x = x, x+y = u$. Preslikavanje $(x,y) \to (x,u)$ oblast $(-\pi,\pi) \times (-\pi,\pi)$ slika u oblast Γ. Neka je $A(N) = A$, niz određen Definicijom 2.1. Izučićemo asimptotsko ponašanje integrala u oblasti $\Gamma_1 \subset \Gamma$ koja je određena uslovom $|u| < \frac{A}{N}$.

$$I_2' = \frac{1}{T^2} \iint_{\Gamma_1} f(x-\lambda_1) f(u-x+\lambda_1) \varphi_M(x) \varphi_M(-x+\lambda_1-\lambda_2) \times$$

$$\times \varphi_M(u-x) \varphi_M(-u+x-\lambda_1+\lambda_2) \frac{\sin^2 \frac{TLu}{2}}{\sin^2 \frac{Lu}{2}} dx du.$$

Primijetimo da je

$$f(x-\lambda_1) f(u-x+\lambda_1) = f(x-\lambda_1)(f(u-x+\lambda_1) - f(\lambda_1)) +$$

$$+ f(\lambda_1)(f(x-\lambda_1) - f(\lambda_2)) + f(\lambda_1) f(\lambda_2). \tag{3.4}$$

Kako je prvi izvod spektralne gustine ravnomjerno ograničen, imamo $|f(u-x+\lambda_1) - f(\lambda_1)| \leq C_1 |u-x|$. Koristeći razlaganje (3.2), integral I_2' zapisujemo u obliku sume integrala I_{21}', I_{22}' i I_{23}'. Sada imamo

$$|I_{21}'| \leq \frac{1}{T^2} \int_{|u|<\frac{A}{N}} \frac{\sin^2 \frac{TLu}{2}}{\sin^2 \frac{Lu}{2}} du \Big(\int_{-\pi}^{\pi} |x|^2 |\varphi_M(x)|^2 |\varphi_M(x+\lambda_1-\lambda_2)|^2 dx \Big)^{\frac{1}{2}} \times$$

$$\times \Big(\int_{-\pi}^{\pi} |\varphi_M(x)|^2 |\varphi_M(x+\lambda_1-\lambda_2)|^2 dx \Big)^{\frac{1}{2}} = O\Big(\frac{1}{N} \Big(\int_{-\pi}^{\pi} |x|^4 |\varphi_M(x)|^4 dx \Big)^{\frac{1}{4}} \times$$

$$\times \Big(\int_{-\pi}^{\pi} |\varphi_M(x)|^4 dx \Big)^{\frac{3}{4}} \Big) = o\left(\frac{1}{N} \int_{-\pi}^{\pi} |\varphi_M(x)|^4 dx \right),$$

ravnomjerno po λ_1 i λ_2.

Nadalje,

$$|f(x-\lambda_1)-f(\lambda_2)| = |f(x-\lambda_1+\lambda_2-\lambda_2)-f(-\lambda_2)| \le C_1|x-(\lambda_1-\lambda_2)|,$$

pa je

$$|I'_{22}| \le \frac{f(\lambda_1)}{T^2}\iint\limits_{\Gamma_1} |x||\varphi_M(x)||\varphi_M(-x+\lambda_1-\lambda_2)||\varphi_M(u-x)|\times$$

$$\times|\varphi_M(-u+x-\lambda_1+\lambda_2)|\frac{\sin^2\frac{TLu}{2}}{\sin^2\frac{Lu}{2}}dxdu+\frac{f(\lambda_1)}{T^2}|\lambda_1-\lambda_2|^*\iint\limits_{\Gamma_1}|\varphi_M(x)\times$$

$$\times\varphi_M(-x+\lambda_1-\lambda_2)\varphi_M(u-x)\varphi(-u+x-\lambda_1+\lambda_2)|\frac{\sin^2\frac{TLu}{2}}{\sin^2\frac{Lu}{2}}dxdu. \quad (3.5)$$

Istovjetnim postupkom kao kod ocjene integrala I'_{21}, pokazuje se da prvi član u sumi (3.3) ima asimptotiku $o\Big(N^{-1}\int\limits_{-\pi}^{\pi}|\varphi_M(x)|^4dx\Big)$ koja je ravnomjerna po λ_1 i λ_2.

U dokazu naše teoreme dva puta će se pojaviti integral

$$J=\frac{1}{T^2}\iint\limits_{\Gamma_1}\frac{\sin^2\frac{TLu}{2}}{\sin^2\frac{Lu}{2}}|\varphi_M(x)\varphi_M(-x+\lambda_1-\lambda_2)\times$$

$$\times\varphi_M(-u+x-(\lambda_1-\lambda_2))\varphi_M(u-x)|dxdu.$$

Izučimo njegovo asimptotsko ponašanje. Uvođenjem smjene $Lu=v$ i korišćenjem Leme 2.2 dobijamo

$$J=\frac{2\pi}{N}\int\limits_{|v|\le\frac{AL}{N}}\frac{1}{2\pi T}\frac{\sin^2\frac{Tv}{2}}{\sin^2\frac{v}{2}}dv\int\limits_{-\pi}^{\pi}|\varphi_M(x)|^2|\varphi_M(x+\lambda_1-\lambda_2)|^2dx+$$

$$+o\left(\frac{1}{N}\int\limits_{-\pi}^{\pi}|\varphi_M(x)|^4dx\right),$$

ravnomjerno po λ_1 i λ_2.

Posebno izučimo

$$J^*=\frac{2\pi}{N}\int\limits_{|v|\le\frac{AL}{N}}\frac{1}{2\pi T}\frac{\sin^2\frac{Tv}{2}}{\sin^2\frac{v}{2}}dv=\frac{2\pi}{N}\Big(1-\int\limits_{\frac{AL}{N}\le|v|\le\pi}\frac{1}{2\pi T}\frac{\sin^2\frac{Tv}{2}}{\sin^2\frac{v}{2}}dv\Big).$$

Korišćenjem nejednakosti $|\sin\frac{x}{2}| \geq \frac{|x|}{2}$, $|x| \leq \pi$, dobijamo

$$\int\limits_{\frac{AL}{N} \leq |v| \leq \pi} \frac{1}{2\pi T} \frac{\sin^2 \frac{Tv}{2}}{\sin^2 \frac{v}{2}} dv \leq \int\limits_{\frac{AL}{N} \leq |v| \leq \pi} \frac{1}{2\pi T} \frac{\pi^2}{v^2} dv = \frac{\pi}{T} \int\limits_{\frac{AL}{N}}^{\pi} \frac{dv}{v^2} =$$

$$= \frac{\pi}{T}\Big(\frac{N}{AL} - \frac{1}{\pi}\Big) \leq \frac{\pi}{T} \cdot \frac{N}{AL} = \frac{\pi}{A} = o(1).$$

Znači,

$$J = \frac{2\pi}{N}(1 - o(1)) \int_{-\pi}^{\pi} |\varphi_M(x)|^2 |\varphi_M(x + \lambda_1 - \lambda_2)|^2 dx + o\left(\frac{1}{N}\int_{-\pi}^{\pi} |\varphi_M(x)|^4 dx\right) =$$

$$= \frac{2\pi}{N}\int_{-\pi}^{\pi} |\varphi_M(x)|^2 |\varphi_M(x + \lambda_1 - \lambda_2)|^2 dx + o\left(\frac{1}{N}\int_{-\pi}^{\pi} |\varphi_M(x)|^4 dx\right), \quad (3.6)$$

ravnomjerno po λ_1 i λ_2, pa je konačno iz (3.3) i (3.4)

$$I'_{22} = |\lambda_1 - \lambda_2|^* O\Big(\frac{1}{N}\int_{-\pi}^{\pi} |\varphi_M(x)|^2 |\varphi_M(x + \lambda_1 - \lambda_2)|^2 dx\Big) + o\left(\frac{1}{N}\int_{-\pi}^{\pi} |\varphi_M(x)|^4 dx\right),$$

ravnomjerno po λ_1 i λ_2.

$$I'_{23} = \frac{f(\lambda_1)f(\lambda_2)}{T^2} \iint\limits_{\Gamma_1} \varphi_M(x)\varphi_M(-x + \lambda_1 - \lambda_2) \cdot$$

$$\cdot \varphi_M(-u + x - \lambda_1 + \lambda_2) \frac{\sin^2 \frac{TLu}{2}}{\sin^2 \frac{Lu}{2}} dx du.$$

Na osnovu Leme 2.1 dobijamo da je

$$I'_{23} = \frac{2\pi}{N} f(\lambda_1) f(\lambda_2) \int_{-\pi}^{\pi} |\varphi_M(x)|^2 |\varphi_M(x + \lambda_1 - \lambda_2)|^2 dx + o\left(\frac{1}{N}\int_{-\pi}^{\pi} |\varphi_M(x)|^4 dx\right),$$

ravnomjerno po λ_1 i λ_2.

Označimo sa $\Gamma_2 \subset \Gamma$ oblast u kojoj je $\frac{A}{N} < u < \frac{\pi}{L}$ i neka je

$$I''_2 = \frac{1}{T^2} \iint\limits_{\Gamma_2} f(x - \lambda_1) f(u - x + \lambda_1) \varphi_M(x) \varphi_M(-x + \lambda_1 - \lambda_2) \times$$

$$\times\varphi_M(u-x)\varphi_M(-u+x-\lambda_1+\lambda_2)\frac{\sin^2\frac{TLu}{2}}{\sin^2\frac{Lu}{2}}dxdu.$$

Sada je

$$|I_2''|\le\frac{D}{T^2}\int\limits_{\frac{A}{N}<|u|<\frac{\pi}{L}}\frac{\sin^2\frac{TLu}{2}}{\sin^2\frac{Lu}{2}}\left(\int\limits_{-\pi}^{\pi}|\varphi_M(x)|^2|\varphi_M(x+\lambda_1-\lambda_2)|^2dx\right)^{\frac{1}{2}}\times$$

$$\times\left(\int\limits_{-\pi}^{\pi}|\varphi_M(u-x)|^2|\varphi_M(u-x+\lambda_1-\lambda_2)|^2dx\right)^{\frac{1}{2}}du\le\frac{D}{T^2}\int\limits_{-\pi}^{\pi}|\varphi_M(x)|^4dx\times$$

$$\times\int\limits_{\frac{A}{N}}^{\frac{\pi}{L}}\frac{2\pi^2}{(Lu^2)}du=O\left(N\frac{\int_{-\pi}^{\pi}|\varphi_M(x)|^4dx}{L^2T^2A}\right)=o\left(\frac{1}{N}\int\limits_{-\pi}^{\pi}|\varphi_M(x)|^4dx\right),$$

gdje je D konstanta.

Ocijenimo integral u oblasti u kojoj je $|u|>\frac{\pi}{L}$. Zbog jednakosti integrala po oblastima $V_1=\{-x<y<x,\ 0\le y\le\pi\}$, $V_2=\{-x<y<x,\ 0\le x\le\pi\}$, $V_3=\{-x<y<x,\ -\pi\le y\le 0\}$, $V_4=\{x\le y\le -x,\ -\pi\le x\le 0\}$, dovoljno je ocijeniti integral, recimo na V_1. Raslojimo sliku oblasti V_1 u ravni (x,u) pravima $u=\frac{\pi}{L},\ldots,\frac{2k+1}{L}\pi,\frac{2k+3}{L}\pi,\ldots$ i nađimo originale ovih slojeva. Sloj iz V_1 koji je omeđen pravima $y_1=-x+\frac{2k+1}{L}\pi$ i $y_2=-x+\frac{2k+3}{L}\pi$ preslikava se u sloj koji je omeđen pravima $u_1=\frac{2k+1}{L}\pi$ i $u_2=\frac{2k+3}{L}\pi$.

Nastavimo sa analizom.

Neka je

$$I_2'''=\frac{1}{T^2}\iint\limits_{|u|>\frac{\pi}{L}}f(x-\lambda_1)f(u-x+\lambda_1)\varphi_M(x)\varphi_M(-x+\lambda_1-\lambda_2)\times$$

$$\times\varphi_M(u-x)\varphi_M(-u+x-\lambda_1+\lambda_2)\frac{\sin^2\frac{TLu}{2}}{\sin^2\frac{Lu}{2}}dxdu$$

i neka je $\Gamma_3=\{2\pi>u>\frac{\pi}{L},\ 2x<u<x+\pi\}$.

Imamo

$$|I_2'''|\le\frac{\bar{C}}{T^2}\iint\limits_{\Gamma_3}\frac{\sin^2\frac{TLu}{2}}{\sin^2\frac{Lu}{2}}|\varphi_M(x)||\varphi_M(-x+\lambda_1-\lambda_2)||\varphi_M(u-x)|\times$$

$$\times|\varphi_M(-u+x-\lambda_1+\lambda_2)|dxdu \leq \frac{\bar{C}}{T}\Big(\int_{-\pi}^{\pi}|\varphi_M(x)|^2dx\Big)\times$$

$$\times\sum_{1\leq k\leq 2L}\int_{\frac{k\pi}{L}}^{\frac{k+2}{L}\pi}\frac{\sin^2\frac{TLu}{2}}{2\pi T\sin^2\frac{Lu}{2}}\Big(\sup_{\frac{k\pi}{2L}<y<\pi}|\varphi_M(y)||\varphi_M(y+\lambda_1-\lambda_2)|\Big)du \leq$$

$$\leq \frac{\bar{C}}{T}\int_{-\frac{\pi}{L}}^{\frac{\pi}{L}}\frac{\sin^2\frac{TLu}{2}}{2\pi T\sin^2\frac{Lu}{2}}du\sum_{1\leq k\leq 2L}\sup_{\frac{k\pi}{2L}<y<\pi}|\varphi_M(y)||\varphi_M(y+\lambda_1-\lambda_2)|,$$

gdje je $\bar{C}$ konstanta.

Kako spektralno jezgro pripada klasi $\mathcal{F}^*$, to je

$$\sum_{1\leq k\leq 2L}\sup_{\frac{k\pi}{2L}<y<\pi}|\varphi_M(y)||\varphi_M(y+\lambda_1-\lambda_2)| = O(L),$$

ravnomjerno po λ_1 i λ_2, te je

$$I_2''' = O\left(LN^{-1}\right) = o\left(\frac{1}{N}\int_{-\pi}^{\pi}|\varphi_M(x)|^4dx\right),$$

ravnomjerno po λ_1 i λ_2.

Znači,

$$I_2 = \frac{2\pi}{N}f(\lambda_1)f(\lambda_2)\int_{-\pi}^{\pi}|\varphi_M(x)|^2|\varphi_M(x+\lambda_1-\lambda_2)|^2dx+$$

$$+|\lambda_1-\lambda_2|^*O\Big(\frac{1}{N}\int_{-\pi}^{\pi}|\varphi_M(x)|^2|\varphi_M(x+\lambda_1-\lambda_2)|^2dx\Big)+$$

$$o\left(\frac{1}{N}\int_{-\pi}^{\pi}|\varphi_M(x)|^4dx\right), \tag{3.7}$$

kada $N\to\infty$, ravnomjerno po λ_1 i λ_2.

Ostaje još da izučimo asimptotiku integrala

$$I_3 = \frac{1}{T^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}f(x-\lambda_1)f(y+\lambda_1)\varphi_M(x)\varphi_M(-x+\lambda_1+\lambda_2)\times$$

$$\times\varphi_M(y)\varphi_M(-y-\lambda_1-\lambda_2)\frac{\sin^2\frac{TL(x+y)}{2}}{\sin^2\frac{L(x+y)}{2}}dxdy.$$

Uvođenjem smjena $x=x, u=x+y$, dobijamo

$$I_3=\frac{1}{T^2}\iint\limits_{\Gamma} f(x-\lambda_1)f(u-x+\lambda_1)\varphi_M(x)\varphi_M(-x+\lambda_1+\lambda_2)\times$$

$$\times\varphi_M(u-x)\varphi_M(-u+x-\lambda_1+\lambda_2)\frac{\sin^2\frac{TLu}{2}}{\sin^2\frac{Lu}{2}}dxdu.$$

Ponavljanjem postupka koji je izložen kod analize integrala I_2, uz korišćenje ocjene

$$|f(x-\lambda_1)-f(\lambda_2)|=|f(x-\lambda_1-\lambda_2+\lambda_2)-f(\lambda_2)|\leq C_1|x-(\lambda_1+\lambda_2)|$$

dobijamo

$$I_3=\frac{2\pi}{N}f(\lambda_1)f(\lambda_2)\int\limits_{-\pi}^{\pi}|\varphi_M(x)|^2|\varphi_M(x+\lambda_1+\lambda_2)|^2dx+$$

$$+|\lambda_1+\lambda_2|^*\times O\Big(\frac{1}{N}\int\limits_{-\pi}^{\pi}|\varphi_M(x)|^2|\varphi_M(x+\lambda_1+\lambda_2)|^2dx\Big)+$$

$$+o\left(\frac{1}{N}\int\limits_{-\pi}^{\pi}|\varphi_M(x)|^4dx\right), \tag{3.8}$$

kada $N\to\infty$, ravnomjerno po λ_1 i λ_2.

Objedinjujući asimptotske jednakosti (3.1), (3.5) i (3.6) dobijamo tvrdjenje teoreme.

Dokaz Teoreme 2.2. Kako je $|\varphi_M(x)|^2=G_M(x)+\epsilon_M(x)$, gdje je $\epsilon_M(x)$ funkcionalni niz koji uniformno konvergira ka 0, dobijamo iz Teoreme 2.1

$$cov(\bar{f}_N(\lambda_1),\bar{f}_N(\lambda_2))=\frac{2\pi}{N}f(\lambda_1)f(\lambda_2)\int\limits_{-\pi}^{\pi}G_M(x)(G_M(x+\lambda_1-\lambda_2)+$$

$$+G_M(x+\lambda_1+\lambda_2))dx+|\lambda_1-\lambda_2|^*O\Big(\frac{1}{N}\int\limits_{-\pi}^{\pi}G_M(x)G_M(x+\lambda_1-\lambda_2)dx\Big)+$$

$$+|\lambda_1+\lambda_2|^* O\Big(\frac{1}{N}\int_{-\pi}^{\pi} G_M(x)G_M(x+\lambda_1+\lambda_2)dx\Big)+o\Big(\frac{1}{N}\int_{-\pi}^{\pi} G_M^2(x)dx\Big)+$$

$$+\frac{2\pi}{N}f(\lambda_1)f(\lambda_2)\Big[\int_{-\pi}^{\pi}\epsilon_M(x)G_M(x+\lambda_1-\lambda_2)dx+\int_{-\pi}^{\pi}\epsilon_M(x)\epsilon_M(x+\lambda_1-\lambda_2)dx+$$

$$+\int_{-\pi}^{\pi}\epsilon_M(x+\lambda_1-\lambda_2)G_M(x)dx+\int_{-\pi}^{\pi}\epsilon_M(x)G_M(x+\lambda_1+\lambda_2)dx+$$

$$+\int_{-\pi}^{\pi}\epsilon_M(x+\lambda_1+\lambda_2)G_M(x)dx+\int_{-\pi}^{\pi}\epsilon_M(x)\epsilon_M(x+\lambda_1+\lambda_2)dx\Big]+O\left(N^{-1}\right)=$$

$$=\frac{2\pi}{N}f(\lambda_1)f(\lambda_2)\int_{-\pi}^{\pi}G_M(x)(G_M(x+\lambda_1-\lambda_2)+G_M(x+\lambda_1+\lambda_2))dx+$$

$$+|\lambda_1-\lambda_2|^* O\Big(\frac{1}{N}\int_{-\pi}^{\pi} G_M(x)G_M(x+\lambda_1-\lambda_2)dx\Big)+$$

$$+|\lambda_1+\lambda_2|^* O\Big(\frac{1}{N}\int_{-\pi}^{\pi} G_M(x)G_M(x+\lambda_1+\lambda_2)dx\Big)+$$

$$+o\Big(\frac{1}{N}\int_{-\pi}^{\pi} G_M^2(x)dx\Big)+O\left(N^{-1}\right),$$

čime je dokaz završen.

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