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Stability in a neighborhood of a singular point of a nonlinear mapping

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Let X and Y be Banach spaces. Consider a smooth mapping $F : X \to Y$. Let $x^* \in X$ be given.

Let $y^* = F(x^*) \in Y$ and assume that the mapping F is twice Frechet-differentiable at x^*. In this paper we are concerned with the following questions: for which $(x, y) \in X \times Y$ close to (x^*, y^*) and under which assumption $\operatorname{dist}(x, F^{-1}(y))$ can be estimated from above via the constraints residual $\|F(x) - y\|$.

The answer to this question is well-known if the following regularity condition is satisfied

$$Im\ F'(x^*) = Y, \tag{1}$$

where $Im\ A$ is the image space of a linear operator A. Under this assumption, there exists a constant $c > 0$ such that the following linear estimate

$$dist(x, F^{-1}(y)) \leq c\|F(x) - y\| \tag{2}$$

holds for all (x, y) close enough to (x^*, y^*) [1].

Estimate (2) serves as a motivation for very important concept of metric regularity [1]. Namely, estimate (2) is actually the definition of metric regularity. Under certain smoothness assumption the converse implication is true and thus

$$(1) \Longleftrightarrow (2)$$

It is important to point out that the notion of metric regularity unifies some classical results such as the inverse mapping theorem, the tangent cone theorems and the error bound result.

In this work, we are interested in the cases when condition (1) does not hold, that is

$$Im\, F'(x^*) \neq Y. \tag{3}$$

This is the nonregular or abnormal case, when the linear estimate (2) cannot be guaranteed. Accordingly, the above-mentioned classical results fail in this case. In this paper, we suggest a new estimate for $dist(x, F^{-1}(y))$. Furthermore, from this estimate we derive the inverse function theorem, the tangent cone theorem, and the error bound result, which can be regarded as an extension and further development of the classical results to the nonregular case. These developments rely on the following

Definition 1. *The mapping F is said to be 2-regular at the point x^* in a direction $h \in X$ if the following equality holds:*

$$Im\ F'(x^*) + F"(x^*)[h, Ker\ F'(x^*)] = Y. \tag{4}$$

If F is regular at x^* ($Im\ F'(x^*) = Y$), then evidently F is 2-regular at x^* in any direction $h \in X$, but not vice versa.

Define the following objects:

$$\theta(p) = \frac{p}{\|p\|} \text{ for } p \in Y,\ \theta(0) = 0,$$

$$b(p) = \inf\{\|x\| : F'(x^*)x = \theta(p)\}\ \forall p \in Im\ F'(x^*).$$

According to the Banach theorem, if the image space of the first derivative is closed, then there exists a constant a such that the inequality $b(p) \le a$ holds for all $p \in Im\ F'(x^*)$. Define the following sets:

$$Cone(\varepsilon, h) = \{x \in X |\ \exists \lambda > 0 : \|x/\lambda - h\| \le \varepsilon\},$$

Then the following stability theorem is valid:

Theorem. (stability theorem) *Let F be 2-regular at x^* in a direction $h \in X$.*

Then there exist $c = c(h) > 0$ and $\varepsilon = \varepsilon(h) > 0$ such that for all $p \in Im\,F'(x^*)$, the following estimate

$$dist(x, F^{-1}(y)) \le c\left(b(p)\|p\| + \frac{\|F(x) - y - p\|}{\|x - x^*\|}\right) \tag{5}$$

holds for all (x, y) close enough to (x^*, y^*) and satisfying the inclusions

$$x \in x^* + Cone(\varepsilon, h),\ \|x - x^*\| \ge c\|F(x) - y - p\|^{1/2}. \tag{6}$$

In the regular case (when $Im\ F'(x^*) = Y$), estimate (5) reduces to linear estimate (2). Indeed, since $Im\ F'(x^*) = Y$, then $\exists a > 0 : b(p) \le a$, $\forall p \in Y$

and we immediately obtain (2) holds for all (x, y) close enough to (x^*, y^*) by setting $p = F(x) - y$, $h = 0$.

Estimate (5) implies some simpler corollaries. Specifically, by the second condition in (6) we obtain the following linear square-root estimate:

$$dist(x, F^{-1}(y)) \leq c\left(b(p)\|p\| + \|F(x) - y - p\|^{1/2}\right) \tag{7}$$

Furthermore, by setting $p = 0$ in (7), we end up with the following square-root estimate

$$dist(x, F^{-1}(y)) \leq \|F(x) - y\|^{1/2},$$
$$\forall (x, y) : x \in x^* + Cone(\varepsilon, h),\ \|x - x^*\| \geq \|F(x) - y\|^{1/2}$$

which is most attractive for applications.

Similar to the regular case, the stated theorem implies three important corollaries: the inverse function theorem, the characterization of a tangent cone, and the error bound.

Define the following set:

$$H(x*) = \{h \in X : F'(x^*)h = 0, F\text{''}(x^*)[h, h] \in cl(Im\, F'(x^*))\}.$$

Corollary 1. (inverse function theorem) *Let F be 2-regular at x^* in a direction $h \in H(x^*)$.*

Then there exist $c = c(h) > 0$ such that for all $p \in Im\ F'(x^*)$ and all y closed enough to y^*, there exists $x(y) \in X$ such that the following relations hold:

$$F(x(y)) = y, \tag{8}$$

$$\|x(y)x^*\| \leq c\left(b(p)\|y - y^*\| + \|y - y^*\|^{1/2}\|\theta(y - y^*) - \theta(p)\|^{1/2}\right). \tag{9}$$

As before, it is easy to see that in the regular case, Corollary 1 reduces $(p = y - y^*,\ h = 0)$ to the classical inverse function theorem.

With $p = 0$, (9) turns into the simple square-root estimate

$$\|x(y) - x^*\| \leq c\|y - y^*\|^{1/2}.$$

For the case when $Im\ F'(x^*)$ is closed then the assertion of Corollary 1 was obtained by author in 1990 [2]. The study of the case of non-closed $Im\ F'(x^*)$ was initiated by H. Sussmann in 2003 [3]. The results of this work in particular imply the existence of the inverse function (that is, condition (8)) in the 2-regular case. The complete version (8)–(9) of this result was obtained in 2005 (in joint work [4]) by E. Avakov and A. Arutyunov

I proceed with the tangent cone result. Define the following sets:

$$H_0(x^*) = \{h \in H(x^*) : (4) \text{ holds } \},$$

$$M(x^*) = \{x \in X : F(x) = F(x^*)\},$$

and let $TM(x^*) = \{h \in X : dist(x^* + th,\, M(x^*)) = o(t)\}$ be the tangent cone to $M(x^*)$ at x^*.

Corollary 2. (tangent cone theorem) *The following inclusions are valid:*

$$H_0(x^*) \subseteq TM(x^*) \subseteq H(x^*) \tag{10}$$

It can be easily seen that in the regular case inclusions (10) give the classical characterization of the tangent cone, known as Lyusternik theorem. Specifically, in this case the tangent cone coincides with the null-space of the first derivative: $TM(x^*) = Ker\ F'(x^*)$.

For the case of closed image space $mathopIm\, F'(x^*)$, relation (10) was obtained in 1985 [5]. For the case of non-closed image space $Im\ F'(x^*)$, relation (10) was obtained in 2005 (in joint work [4]) by E. Avakov and A. Arutyunov.

Definition 2. *The mapping F is said to be 2-regular at the point x^* if it is 2-regular at this point in any direction $h \in H(x^*)\backslash\{0\}$.*

From (10) we immediately obtain, that if F is 2-regular at x^*, then the tangent cone coincides with $H(x^*) : TM(x^*) = H(x^*)$.

Furthermore, define the liner operator $G(x^*, h) : X \times X \to Y$,

$$G(x^*, h)(x_1, x_2) = F'(x^*)x_1 + F\text{''}(x^*)[h, x2].$$

which is 2-regular: $G(x^*, h)(X \times Ker\, F'(x^*)) = Y\ \forall h \in H(x^*)\backslash\{0\}$.

Definition 3. *The mapping F is said to be strongly 2-regular at the point x^* if exist $c > 0$ and $\varepsilon > 0$ such that for all $y \in Y\, \exists (x_1, x_2) \in X \times Ker\ F'(x^*) : G(x^*, h)(x_1, x_2) = y$, $\|(x_1, x_2)\| \le c\|y\|$ for all $(x, h) \in X \times X$, $\|h\| = 1$, $\|F'(x^*)h\| \le \varepsilon$, $G(x^*, h)(x, h)\| \le \varepsilon$.*

If $dim\,(X) < \infty$ and $dim\,(Y) < \infty$, then Definition 2 (2-regularity) and Definition 3 (strongly 2-regularity) are equivalent.

Corollary 3. (error bound) *Let F be strongly 2-regular at x^*.*

Then there exists $c = c(x^*) > 0$ such that for any $p \in Im\ F'(x^*)$, the following estimate

$$dist\,(x, (x^*)) \le c\left(b(p)\|p\| + \frac{\|F(x) - F(x^*) - p\|}{\|x - x^*\|}\right), \tag{11}$$

holds for all x *close enough to* x^*.

As above, in the regular case (when $Im\ F'(x^*) = Y$) estimate (11) reduces ($p = F(x) - F(x^*), h = 0$) to the well-known linear error bound:

$$dist\,(x, (x^*)) \leq \|F(x) - F(x^*)\|.$$

For the case when $Im\ F'(x^*)$ is closed then the assertion of Corollary 3 was obtained by author in 1990, [2].

The following simple corollary of estimate (11) may be more convenient for applications (square-root error bound):

$$dist\,(x, (x^*)) \leq \|F(x) - F(x^*)\|^{1/2}.$$

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