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ON THE WELL-POSEDNESS OF QUADRATIC PROGRAMMING PROBLEMS IN HILBERT SPACE

A b s t r a c t

In this paper we will present some necessary and sufficient conditions for well-posedness of quadratic minimization problem with linear and quadratic constraints.

KOREKTNOST ZADATKA KVADRATNOG PROGRAMIRANJA U HILBERTOVOM PROSTORU

I z v o d

U radu su dati neophodni i dovoljni uslovi korektnosti zadatka minimizacije kvadratnog funkcionala sa linearnim i kvadratnim ograničenjima.

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1 Introduction

An optimization problem is well-posed if its set of solutions attracts all approximate solutions corresponding to small perturbations of the given problem. This statement can be formalized in a different way.

According to Hadamard's concepts of well-posedness, the optimization problem $\inf\{J(u) : u \in U\}$ is well-posed if $J : U \mapsto \mathbb{R}$ has a unique point of minimum on metric space U depending continuously on the data J and U.

Zolezzi (s. [1], [8], [9]) considers the well-posedness of the problem $\inf\{J(u) : u \in U\}$, by perturbations, defined by parameter $p \in L \subseteq P$ and function $(u, p) \mapsto F(u, p)$, $(u, p) \in (U, L)$, where P is a normed space, $L = \{p \in P : \|p - p_*\| \leq r\}$ is the closed ball of center p_* and positive radius r, and $F(u, p_*) = J(u)$.

According to Zolezzi's definition, the problem $\inf\{J(u) : u \in U\}$ is wellposed if $V(p) := \inf\{F(u, p) : u \in U\} > -\infty$ for every $p \in L$, its set of solutions $U_* = \{u \in U : J(u) = V(p_*)\}$ is nonempty, and for every sequences $(p_n), p_n \in L$ and $(u_n), u_n \in U$ such that $F(u_n, p_n) - V(p_n) \to V(p_*)$, there exists an subsequence (u_{n_k}) which converges to U_*.

The main interests of our investigations are related to Tikhonov's well-posedness.

Problem of minimization of function J on metric space U is said to be well posed according to Tikhonov if the following conditions are satisfied:

(i) $J_* := \inf\{J(u) : u \in U\} > -\infty$,

(ii) $U_* := \{u \in U : J(u) = J_*\} \neq \emptyset$,

(iii) for every sequence (u_n) from U such that $J(u_n) \to J_*$, $d(U_*, u_n) := \inf\{d(u_n, u) : u \in U_*\} \to 0$ as $n \to \infty$

The sequence (u_n) which satisfies the condition (iii) is said to be minimizing for the minimization problem $J(u) \to \inf$, $u \in U$.

In this paper, the function J is given by

$$J(u) = \|Au - f\|^2 \to \inf, u \in U, \tag{1.1}$$

where $U \subseteq H$ is a closed convex set in a Hilbert space H, $A : H \to F$ is a linear continuous operator from H to Hilbert space F, and $f \in F$ is a fixed element.

The minimization problem of strongly convex function on convex closed subset of Hilbert space is (Tikhonov) well-posed, as it is well known. Function J in (1.1) is convex but it is not necessarily strongly convex. Therefore, the existence of solution of (1.1) and well-posedness of the same problem, are not trivial, even if the set U has very simple structure.

We will investigate the existence and well-posedness of this problem, assuming that the set U is given by one linear and one quadratic constraint:

$$U = U_1 \cap U_2,\ U_1 := \{u \in H : \|Bu\| \leq r\},\ U_2 := \{u \in H : Cu \leq \beta\}. \tag{1.2}$$

Here, $B : H \to G$ is a linear bounded operator from H to Hilbert space G; $Cu = \langle c, u \rangle$ is a linear continuous functional defined on H; $r > 0$ and β given real numbers.

Our purpose is to find some sufficient and/or necessary conditions of existence and well-posedness of problem (1.1), (1.2).

Let us emphasize that all our results regarding well-posedness were obtained under the assumption that all initial data is exactly known; well-posedness related to inexact initial data will not be considered here.

2. AUXILIARY RESULTS

Let us introduce the following notation: $\mathcal{L}(\mathcal{M})-$ the linear hull of the set $M \subseteq H$, $I-$ the identity operator, $R(A)$ - the range of the operator A, $A(U) = \{Au : u \in U\}$, $Ker\, A-$ the null-space of A, $\overline{M}-$ the closure of the set $M \subseteq H$, $L^{\perp}-$ the orthogonal complement of the subspace L, $P-$ the orthogonal projection operator from H to

$\overline{R(A^*)}$, $Q-$ the orthogonal projection operator on H to $\overline{R(B^*)}$, P_r- the orthogonal projection operator on F to the closed and convex set $\overline{A(U)}$, B_1 - the restrictions of the operators B to the subspace $Ker\,A \cap Ker\,C$ and A_1 - the restriction of the operator A to the subspace $Ker\,B \cap Ker\,C$, A_B - the restriction of the operator A to the subspace $Ker\,B$, and $S = \{u \in H : \|Bu\| = r, \langle c, u\rangle = \beta\}$ - the intersection of the boundaries of the elipsoid U_1 and the half-space U_2.

The operator A produces the following orthogonal decompositions of the spaces H and F :

$$H = \overline{R(A^*)} \oplus Ker\,A, \; F = \overline{R(A)} \oplus Ker\,A^*. \tag{2.3}$$

Further, the next decomposition holds for any two closed subspaces L and M, of a Hilbert space H :

$$(L \cap M)^{\perp} = \overline{L^{\perp} + M\perp}, \; H = \overline{L^{\perp} + M\perp} \oplus (L \cap M). \tag{2.4}$$

Lemma 2.1 *For the operators A, B and C the following decompositions are true:*

$$H = \overline{R(A^*)} \oplus \mathcal{L}((I-P)c) \oplus \overline{R(B_1^*)} \oplus (Ker\,A \cap Ker\,B \cap Ker\,C), \tag{2.5}$$

$$H = \overline{R(B^*)} \oplus \mathcal{L}((I-Q)c) \oplus \overline{R(A_1^*)} \oplus (Ker\,A \cap Ker\,C), \tag{2.6}$$

$$Ker\,B = \overline{R(A_B^*)} \oplus (Ker\,A \cap Ker\,B), \tag{2.7}$$

Proof. Using the decompositions (2.4) and (2.3) we obtain

$$\begin{aligned} H &= \overline{(Ker\,A)^{\perp} \oplus (Ker\,C)^{\perp}} \oplus (Ker\,A \cap Ker\,C) \\ &= \overline{R(A^*)} \oplus \mathcal{L}(c) \oplus (Ker\,A \cap Ker\,C) \\ &= \overline{R(A^*)} \oplus \mathcal{L}((I-P)c) \oplus (Ker\,A \cap Ker\,B \cap Ker\,C). \end{aligned}$$

Similarly, applying (2.3) to $B_1 : Ker\,A \cap Ker\,C \to G$ we obtain

$$Ker\,A \cap Ker\,C = \overline{R(B_1^*)} \oplus (Ker\,A \cap Ker\,B \cap Ker\,C).$$

Hence, we have proved the equality (2.5); (2.6) can be proved in a similar way.

The next lemma is related to normally solvable operators.

An operator $A : H \mapsto F$ is said to be normally solvable if $R(A) = \overline{R(A)}$. Let us remark that this is equivalent with $R(A^*) = \overline{R(A^*)}$. (s. [5], pp. 153.)

Lemma 2.2 ([5], pp. 153) *A bounded linear operator $A : H \to F$ is normally solvable if and only if*

$$\mu := \inf\{\|Au\| : u \perp Ker\, A, \|u\| = 1\} > 0.$$

The immediate consequence of this Lemma is the following.

Lemma 2.3 ([5], pp. 153) *If the linear operator $A : H \to F$ is not normally solvable, then there exists a sequence (p_n) such that $p_n \in \overline{R(A^*)}$, $\|p_n\| = 1$ and $Ap_n \to 0$ as $n \to \infty$.*

The restriction of a normally solvable operator $A : H \to F$ to the subspace $R(A^*)$ is invertible, so there exists $M > 0$ such that

$$(\forall x \in R(A^*))\ \|x\| \leq M\|Ax\|. \tag{2.8}$$

If $A(V)$ is a closed for a closed set $V \subseteq H$, then the inverse image

$$A^{-1}(AV) = Ker\, A + V$$

is closed set. If A is a normally solvable operator, then the converse statement is also true: if $Ker\, A + V$ is a closed set, then the set $A(V)$ is also closed [2].

Now, it is easy to prove that for a normally solvable operator A and for a closed subspace $M \subseteq H$ of a finite codimension, we have $A(M) = \overline{A(M)}$.

Namely, $codim\, M < +\infty$, implies that $dim\, M^{\perp} < +\infty$. Let us denote the operator of orthogonal projection from H to $M^{\perp}$ with T. It is easy to prove the equality $M + Ker\, A = M + T(Ker\, A)$. From $T(Ker\, A) \subseteq M^{\perp}$, it follows that $dim(T(Ker\, A) < +\infty$. So, what we have is that the set $M + Ker\, A$ is closed. Normal solvability of the operator A implies that $A(M) = \overline{A(M)}$.

Lemma 2.4 *Let L and M be closed subspaces of a space H. If $dim\, L < +\infty$, then $A(M) = \overline{A(M)}$ if and only if $A(L \cap M) = \overline{A(L \cap M)}$.*

Proof. From $codim\, L < +\infty$ it follows that there exist $h_1, \ldots, h_n$ in H, such that $L^\perp = \mathcal{L}(h_1, \ldots, h_n)$, i.e.

$$H = \mathcal{L}(h_1, \ldots, h_n) \oplus L.$$

As earlier, let us denote again the operator of orthogonal projection of the space H onto $M^\perp$ with T. Note

$$M^\perp \oplus \mathcal{L}(h_1, \ldots, h_n) \quad = \quad M^\perp \oplus \mathcal{L}((I-T)h_1, \ldots, (I-T)h_n).$$

Applying (2.4) we obtain

$$\begin{aligned} H &= (M^\perp \oplus \mathcal{L}(h_1, \ldots, h_n)) \oplus M \cap L \\ &= M^\perp \oplus \mathcal{L}((I-T)h_1, \ldots, (I-T)h_n) \oplus M \cap L. \end{aligned}$$

This equality and decomposition $H = M \oplus M^\perp$ imply that

$$M = \mathcal{L}((I-T)h_1, \ldots, (I-T)h_n) \oplus (M \cap L). \tag{2.9}$$

If $A(M \cap L) = \overline{A(M \cap L)}$ then, using (2.9), we obtain $A(M) = \overline{A(M)}$. Now, assume that $A(M) = \overline{A(M)}$. It means that the restriction of the operator A to the subspace M is a normally solvable operator. From (2.9), we conclude that $M \cap L$ is a closed subspace of a finite codimension in the subspace M. Hence, $A(L \cap M)$ is a closed subspace of the space H.

Lemma 2.5 *If $Int\, U = \emptyset$, $U = \{u \in H : \|Bu\| \le r, \langle c, u\rangle \le \beta\}$, then*

(i) $U = S := \{u \in H : \|Bu\| = r, \langle c, u\rangle = \beta\}$;
(ii) $Ker\, B \subseteq Ker\, C$, *where* $Cu = \langle c, u\rangle$;
(iii) $(\forall u \in U)\, U = u + Ker\, B$.

Proof. (i) If $\|Bv\| < r$ and $v \in U$ then there exists an open set $V(v)$ containing v, such that $\|Bx\| < r$ for every $x \in V(v)$. In this case, taking into account that $Int\, U = \emptyset$, we can conclude that $\langle c, v\rangle = \beta$. But, then there exists a point $x_0 \in V(v)$ such that $\langle c, x_0\rangle < \beta$. This contradicts $Int\, U = \emptyset$. So, we have $\|Bv\| = r$ for every $v \in U$. Similarly, we can prove that $\langle c, v\rangle = \beta$ for every $v \in U$. Hence, $U = S$.

(ii) We will prove that $(I - Q)c = 0$, where Q is the orthogonal projection onto $\overline{R(B^*)}$. Assume the converse. Then a point $v = u + \gamma(I - Q)c, u \in U = S$, $\gamma < 0$, satisfies the conditions $\|Bv\| = r$ and $\langle c, v\rangle < \beta$. Since $U = S$, we have a contradiction. Hence, $(I - Q)c = 0$, i.e. $c \in R(B^*) \perp Ker\, B$. It immediately implies the inclusion $Ker\, B \subseteq Ker\, C$.

(iii) Let x and u be arbitrary points from $U = S$. Then $\langle c, x-u\rangle = \langle c, x\rangle - \langle c, u\rangle = 0$. Hence, $x - u \in Ker\, C$. Since, $U = S$ is a convex set, it follows that $\|B(\alpha u + (1 - \alpha x)\| = r$ for any $\alpha \in [0, 1]$. This implies that $\langle Bx, Bu\rangle = r^2$. Thus we have $\|B(x - u)\| = r^2 - 2r^2 + r^2 = 0$, i.e. $x - u \in Ker\, B$. So, we proved the inclusion $U \subseteq u + Ker\, B \cap Ker\, C$. The converse inclusion is trivial. Now, (iii) follows from $U = u + Ker\, B \cap Ker\, C$ and (ii).

Lemma 2.6 *If there exists $u \in S$ such that $B^*Bu \in \mathcal{L}(c)$ and $\beta < 0$, then $Int\, U = \emptyset$.*

Proof. Suppose $B^*Bu = \alpha c, \alpha \neq 0$. Multiplying this equality by u, we obtain $r^2 = \|Bu\|^2 = \alpha\langle c, u\rangle = \alpha\beta$. Since $\beta < 0$, it follows that $\alpha < 0$.

Assume that $Int\, U \neq \emptyset$. Then there exists $v \in U$ such that $\|Bv\| < r$ and $\langle c, v\rangle < \beta$. It now follows that $\langle Bu, Bv\rangle \leq \|Bu\| \cdot \|Bv\| < r^2$. We obtained the contradiction that proves Lemma.

3. RESULTS

3.1 Existence of solutions

It is clear that the problem (1.1), (1.2) has a solution if and only if the projection $P_r(f)$ of f on $\overline{A(U)}$ belongs to $A(U)$. Taking into account that $P_r(F) = \overline{A(U)}$, we can conclude that the problem (1.1), (1.2) has a solution for every $f \in F$ if and only if $A(U) = \overline{A(U)}$.

Note that convexity and continuity of the function J imply its lower weakly semi-continuous. The set U is weakly closed, because it is convex and closed. Now, it is easy to prove that if for any $f \in F$ there exists at least one minimizing sequence (u_n), then, for such an f, problem (1.1), (1.2) has a solution.

Namely, then there exist a subsequence (u_{n_k}) of (u_n) and a point $u_* \in H$, so that (u_{n_k}) weakly converges to u_*. Since the set U is weakly closed, $u_* \in U$. This and lower semi-continuous of J imply

$$J(u_*) \leq \liminf J(u_{n_k}) = J_*.$$

Hence, $J(u_*) = J_*$, i.e. $u_* \in U_*$.

Theorem 3.1 *Suppose the following conditions hold:*
(i) A is a normally solvable operator;
(ii) $B(Ker\, A)$ is closed subspace of G..

Then the problem (1.1), (1.2) has a solution for every $f \in F$.

Proof. We will prove that for each $f \in F$ there exists a bounded minimizing sequence. Condition (ii) and Lemma 4 imply that the operator B_1 is normally solvable. Since the operator A is also normally solvable, it follows that the equality (2.5) can be written as

$$H = R(A^*) \oplus \mathcal{L}((I-P)c) \oplus R(B_1^*) \oplus (Ker\, A \cap Ker\, B \cap Ker\, C).$$

The elements of minimizing sequence (u_n) can be decomposed in the following way

$$\begin{aligned} u_n = \quad & Pu_n + \gamma_n(I-P)c + b_n^* + b_n, \\ & \gamma_n \in \mathbb{R}, b_n^* \in R(B_1^*), b_n \in Ker\, A \cap Ker\, B \cap Ker\, C. \end{aligned}$$

Note that the sequence $w_n = Pu_n + \gamma_n(I-P)c + b_n^*$ is also minimizing. Since $Pu_n \in R(A^*)$, $Aw_n = Au_n = APu_n$ and $\|Au_n - f\| \to J_*$ as $n \to \infty$, it follows that the sequence $(B(\gamma_n(I-P)c + b_n^*))$ is bounded.

Then, the boundedness of the sequence $(B(\gamma_n(I-P)c+b_n^*))$ follows from

$$\|B(Pu_n+\gamma_n(I-P)c+b_n^*)\| = \|Bw_n\| \le r.$$

Hence, there exists a constant $k>0$ such that

$$\|B(\gamma_n(I-P)c+b_n^*)\| \le k,\ n=1,2,\ldots,. \tag{3.10}$$

We will consider two possibilities.

(a) Suppose that the sequence (γ_n) is bounded or $(I-P)c=0$. Then the boundedness of the sequence (Bb_n^*) follows from (3.10). Applying (2.8) to the operator B_1, and taking into account that $b_n^* \in R(B_1^*)$, we obtain that the sequence (b_n^*) is also bounded. Hence, in the case of (a), there exists a bounded minimizing sequence for problem (1.1), (1.2).

(b) Now, suppose that the sequence (γ_n) is unbounded and $(I-P)c \neq 0$. Let us prove the following relations:

$$(I-P)c = p_0+z_0,\ \langle z_0,c\rangle \neq 0,\ p_0 \in R(B_1^*),\ z_0 \in Ker\,A \cap Ker\,B. \tag{3.11}$$

With respect to the operator $B_1 : Ker\,A \cap Ker\,C \to G$, the spaces $Ker\,A \cap Ker\,C$ and G can be decomposed as follows

$$Ker\,A \cap Ker\,C = R(B_1^*) \oplus Ker\,B_1,\ G = R(B_1) \oplus Ker\,B_1^*.$$

This implies that for $B(I-P)c \in G$ there exist $p_0 \in R(B_1^*)$ and $q_0 \in Ker\,B_1^*$, such that

$$B(I-P)c = Bp_0 + q_0.$$

Since $B(\gamma_n p_0 + b_n^*) \perp q_0$, we have

$$\begin{aligned}\|B(\gamma_n(I-P)c+b_n^*)\|^2 &= \|B(\gamma_n p_0+b_n^*)+\gamma_n q_0\|^2\\ &= \|B(\gamma_n p_0+b_n^*)\|^2+\gamma_n^2\|q_0\|^2.\end{aligned}$$

The last equality and (3.11) imply $q_0=0$. Then $B((I-P)c-p_0)=0$, and consequently, there is $z_0 \in Ker\,B$, such that

$$(I-P)c = p_0+z_0.$$

Noting that $p_0 \in R(B_1^*) \subseteq Ker\, A \cap Ker\, C$ and $(I-P)c \in Ker\, A$, we infer

$$0 = A(I-P)c = A(p_0 + z_0) = Az_0,$$

i.e. $z_0 \in Ker\, A \cap Ker\, B$. If we take the scalar product of each side of the equality $(I-P)c = p_0 + z_0$ with c, we will have

$$\langle c, z_0 \rangle = \|(I-P)c\|^2 \neq 0,$$

which proves (3.11). Now, we know that

$$B(\gamma_n(I-P)c + b_n^*) = B(\gamma_n p_0 + b_n^*).$$

Therefore, from $\gamma_n p_0 + b_n^* \in R(B_1^*)$ and (3.10), and applying (2.8) to the operator B_1, it follows that $(\gamma_n p_0 + b_n^*)$ is a bounded sequence. Observe the bounded sequence

$$v_n = Pu_n + \gamma_n p_0 + b_n^* + \gamma_n^* z_0, \text{ where } \gamma_n^* = \frac{\beta - \langle Pu_n + \gamma_n p_0 + b_n^*, c\rangle}{\langle c, z_0 \rangle}.$$

It is obvious that

$$Av_n = Aw_n, Bv_n = Bw_n, \langle c, v_n \rangle = \beta,$$

which makes (v_n) a bounded minimizing sequence. This completes the proof.

Similarly, using orthogonal decomposition (2.6), we can prove the following theorem.

Theorem 3.2 *Suppose that:*

(i) B is a normally solvable operator;

(ii) $A(Ker\, B)$ is a closed subspace of H;.

Then, for every $f \in F$ the problem (1.1), (1.2) has a solution.

Now, let us consider the case of $U = \emptyset$.

Theorem 3.3 *Let $Int\, U = \emptyset$. Then the problem (1.1), (1.2) has a solution for every $f \in F$ if and only if $A(Ker\, B)$ is a closed subspace of F.*

Proof. Lemma 5 (iii) implies that for every $u \in U$, we have $A(U) = Au + A(Ker\, B)$ which proves the Theorem.

3.2 Well-posedness

In this section,we will discuss the well-posedness of the problem (1.1), (1.2). Note that if $U_* \neq \emptyset$, then for every $u_* \in U_*$,

$$U_* = (u_* + Ker\, A) \cap U.$$

From $J(u) = J(v) + \langle J'(v), u - v\rangle + \|A(u - v)\|^2$ and from optimality criterion of the element $u_* \in U_*$ (s. [7], p. 161, Theorem 3) $(\forall u \in U)\langle J'(u_*), u - u_*\rangle \geq 0$, we have $\|Au - Au_n\|^2 \leq J(u) - J(u_*)$.

This implies $Au_n \to Au_*$ as $n \to \infty$, for every minimizing sequence (u_n).

If operator A is normally solvable, then, from (2.3) and (2.8) (for operator P of orthogonal projection from H to $\overline{R(A^*)}$) we have

$$\|P(u_n - u_*)\| \leq m\|AP(u_n - u_*)\| = \|Au_n - Au_*\| \to 0 \text{ as } n \to \infty$$

i.e. $Pu_n \to Pu_*$ as $n \to \infty$.

Next theorem shows that the conditions of the Theorem 3.1 guarantee, not only the existence of solution, but also the wellposedness of minimizing sequences of the problem (1.1), (1.2).

Theorem 3.4 *Suppose that*

(i) A is a normally solvable operator;

(ii) $B(Ker\, A)$ is closed subspace of G;.

Then the problem (1.1), (1.2) is well-posed for every $f \in F$.

Proof. Theorem 3.1 implies that$U_* \neq \emptyset$. Just as in the proof of Theorem 1, every minimizing sequence (u_n) can be written as

$$\begin{aligned} u_n = \quad & Pu_n + \gamma_n(I - P)c + b_n^* + b_n, \\ & \gamma_n \in R, b_n^* \in R(B_1^*), b_n \in Ker\, A \cap Ker\, B \cap Ker\, C. \end{aligned}$$

Observe the minimizing sequence

$$w_n = Pu_n + \gamma_n(I - P)c + b_n^*$$

and note that (i) implies

$$Pu_n \to Pu_* \text{ as } n \to \infty.$$

Let us consider two cases.

(a) Suppose that a sequence (γ_n) is bounded or that $(I-P)c \neq 0$. In each case we can assume $\gamma_n \to \gamma_* \in R$ as $n \to \infty$. Since $b_n^* + b_n \perp c$ we have

$$\langle Pu_* + \gamma_*(I-P)c, c\rangle = \lim_{n\to\infty} \langle Pu_n + \gamma_n(I-P)c, c\rangle = \lim_{n\to\infty} \langle w_n, c\rangle \leq \beta$$

The sequence $v_n = Pu_* + \gamma_*(I-P)c + b_n^* + b_n$ satisfies the inequality

$$\langle v_n, c\rangle \leq \beta.$$

(a_1) If $\|Bv_n\| \leq r$, then $v_n \in U_*$, and therefore

$$d(u_n, U_*) \leq \|u_n - v_n\| \leq \|Pu_n - Pu_* + (\gamma_n - \gamma_*)(I-P)c\| \to 0,\ n \to \infty.$$

(a_2) Now, assume that $\|Bv_n\| > r$. Then

$$\begin{aligned} r < \|Bv_n\| &\leq \|B(v_n - w_n)\| + \|Bw_n\| \\ &\leq \|B(Pu_n - Pu_* + (\gamma_n - \gamma_*)(I-P)c)\| + r, \end{aligned}$$

implies

$$\lim_{n\to\infty} \|Bv_n\| = r \text{ and } \lim_{n\to\infty} \|Bw_n\| = r. \tag{3.12}$$

The operator B_1 (restriction of B on $Ker\,A \cap Ker\,C$) is normally solvable. Hence the sequence (b_n^*), $b_n^* \in R(B_1^*)$ is bounded. We can assume that (b_n^*) converges weakly to $b_0^* \in R(B_1^*)$ as $n \to \infty$. Then, the minimizing sequence (w_n) converges weakly to $w_* = Pu_* + \gamma_*(I - P)c + b_0^* \in U_*$.

In scope of this case, we will consider two possibilities:

(a_{21}) If $\|Bw_*\| = r$, then (3.12) together with

$$\|B(b_n^* - b_0^*)\|^2 = \|B(w_n - w_* + Pu_* - Pu_n + (\gamma_n - \gamma_*)(I-P)c\|^2, \tag{3.13}$$

imply $\|B(b_n^* - b_0^*)\| \to 0$ as $n \to \infty$. From $b_n^* - b_0^* \in R(B_1^*)$, applying (2.8) to B_1, it follows that (b_n^*) converges (strongly) to b_0^* as $n \to \infty$. Then $w_n \to w_*$ as $n \to \infty$, and therefore

$$d(u_n, U_*) \leq \|u_n - (w_* + b_n)\| = \|w_n - w_*\| \text{ as } n \to \infty.$$

(a$_{22}$) If $\|Bw_*\| < r$, then (3.12) and (3.13) imply

$$\lim_{n\to\infty} \|B(b_n^* - b_0^*)\|^2 = r^2 - \|Bw_*\|^2 > 0.$$

For each $n \in N$, there exists a number $\alpha_n > 0$ such that $\|B(w_* + \alpha_n(b_n^* - b_0^*))\|^2 = r^2$. Now, using the last two relations, it is easy to prove $\lim_{n\to\infty} \alpha_n = 1$. Sequence

$$x_n = w_* + \alpha_n(b_n^* - b_0^*) + b_n = Pu_* + \gamma_*(I - P)c + \alpha_n b_n^* + (1 - \alpha_n)b_0^* + b_n$$

satisfies the following conditions

$$Ax_n = Aw_*,\ \|Bx_n\| = r, \langle c, x_n\rangle = \langle c, w_*\rangle \leq \beta,$$

and so $x_n \in U_*$ for every $n \in N$. Then

$$\begin{aligned} &d(u_n, U_*) \leq \|u_n - x_n\| \\ &= \|Pu_n - Pu_* + (\gamma_n - \gamma_*)(I - P)c + (1 - \alpha_n)(b_n^* - b_0^*)\| \to 0 \end{aligned}$$

as $n \to \infty$. Therefore, the problem (1.1), (1.2) is well-posed when (a) occurs.

(b) Now, assume the sequence (γ_n) is unbounded and $(I - P)c = 0$. Then, according to (3.11), we can write

$$u_n = Pu_n + \overline{b_n^*} + b_n + \gamma_n z_0 \text{ and } w_n = Pu_n + \overline{b_n^*} + \gamma_n z_0,$$

where

$$\overline{b_n^*} = \gamma_n p_0 + b_n^* \text{ and } \langle z_0, c\rangle \neq 0.$$

Take the sequence

$$y_n = Pu_* + \overline{b_n^*} + b_n + \delta_n z_0$$

where

$$\delta_n = \frac{\langle u_n, c\rangle - \langle Pu_* + \overline{b_n^*}, c\rangle}{\langle z_0, c\rangle} = \frac{\langle Pu_n - Pu_*, c\rangle}{\langle z_0, c\rangle} + \gamma_n,$$

and note that $\lim_{n\to\infty}(\gamma_n - \delta_n) = 0$. The numbers δ_n have been chosen in such way that $\langle y_n, c\rangle \le \beta$.

(b$_1$) If $\|By_n\| \le r$, then $y_n \in U_*$ and therefore

$$d(u_n, U_*) \le \|u_n - y_n\| = \|Pu_n - Pu_* + (\gamma_n - \delta_n)z_0\| \to 1 \text{ as } n \to \infty.$$

(b$_2$) If $\|By_n\| > r$, then following the procedure of (a$_2$) we obtain

$$\lim_{n\to\infty} \|By_n\| = r, \ \lim_{n\to\infty} \|Bw_n\| = r,$$

and

$$\overline{b_n^*} \text{ weakly converges to } \overline{b_0^*} \in R(B_1^*) \text{as } n \to \infty.$$

It follows that

$$Pu_n + \overline{b_n^*} \text{ weakly converges to } \overline{w_*} = \overline{b_0^*} + Pu_* \text{ as } n \to \infty, \text{ and } \|B\overline{w_*}\| \le r.$$

As in case of (a$_2$) we again need to consider two possibilities.

(b$_{21}$) If $\|B\overline{w_*}\| = r$, then, just like in (a$_{21}$), we can prove strong convergence of the sequence $\overline{b_n^*}$ to $\overline{b_0^*}$ as $n \to \infty$. Observe the sequence $z_n = \overline{w_*} + \overline{b_n} + \delta_n^* z_0$ where

$$\delta_n^* = \frac{\langle u_n, c\rangle - \langle \overline{w_*}, c\rangle}{\langle z_0, c\rangle}.$$

Then $\langle z_n, c\rangle \le \beta$ and

$$\delta_n^* - \gamma_n = \frac{\langle Pu_n - Pu_* + \overline{b_n^*} - \overline{b_0^*}, c\rangle}{\langle z_0, c\rangle} \to 0 \text{ as } n \to \infty.$$

Besides, $Az_n = A\overline{w_*}$, $\|Bz_n\| = \|B\overline{w_*}\| = r$, such that $z_n \in U_*$. Now we have

$$d(u_n, U_*) \le \|u_n - z_n\| = \|Pu_n - Pu_* + \overline{b_n^*} - \overline{b_0^*} + (\gamma_n - d_n^*)z_0\| \to 0$$

as $n \to \infty$.

(b_{22}) Finally, let $\|B\overline{w_*}\| < r$. Similarly to (a_{22}) we can prove that

$$\lim_{n\to\infty} \|B(\overline{b_n^*} - \overline{b_0^*})\|^2 = r^2 - \|B\overline{w_*}\|^2 > 0.$$

The numbers $\alpha_n, n = 1, 2, \dots$ are set in such way that

$$\|B(\overline{w_*} + \alpha_n(\overline{b_n^*} - \overline{b_0^*}))\|^2 = r^2 \text{ with } \lim_{n\to\infty} \alpha_n = 1.$$

Take the sequence

$$s_n = \overline{w_*} + \alpha_n(\overline{b_n^*} - \overline{b_0^*}) + b_n + r_n z_0 = Pu_* + \alpha_n \overline{b_n^*} + (1-\alpha_n)\overline{b_0^*} + b_n + \eta_n z_0$$

where

$$\eta_n = \frac{\langle u_n, c\rangle - \langle Pu_* + \alpha_n \overline{b_n^*} + (1-\alpha_n)\overline{b_0^*}, c\rangle}{\langle z_0, c\rangle}.$$

Then $\langle s_n, c\rangle \le \beta$ and

$$\eta_n - \gamma_n = \frac{\langle Pu_n - Pu_* + (1-\alpha_n)(\overline{b_n^*} - \overline{b_0^*}), c\rangle}{\langle z_0, c\rangle} \to 0 \text{ as } n \to \infty.$$

Besides, $As_n = A\overline{w_*}$, $\|Bs_n\| = \|B\overline{w_*}\| = r$, so that $s_n \in U_*$. Therefore

$$d(u_n, U_*) \le \|u_n - s_n\| = \|Pu_n - Pu_* + (1-\alpha_n)(\overline{b_n^*} - \overline{b_0^*}) + (\gamma_n - \eta_n)z_0\| \to 0$$

as $n \to \infty$. This completes the proof.

The next theorem shows that if the first conditions of the previous theorem is violated, then the problem (1.1), (1.2) will not be well-posed anymore.

Theorem 3.5 *Suppose*

(i) $\overline{R(A)} \ne R(A)$;

(ii) $U_* \cap Int\, U_1 \ne \emptyset$.

Then the problem (1.1), (1.2) is not well-posed.

Proof. The condition (i), according to Lemma 2.3, implies the existence of a sequence (p_n) such that

$$p_n \in \overline{R(A^*)},\ \|p_n\| = 1,\ \lim_{n\to\infty} Ap_n = 0.$$

Since $U_* \cap Int\, U_1 \neq \emptyset$, we can infer that there is an element $u_* \in U_*$ such that $\|Bu_*\| < r$. Choose an $\varepsilon_0 > 0$ such that $\|B(u_* \pm \varepsilon_0 p_n)\| < r$. Consider the sequence (v_n):

$$v_n = \begin{cases} u_* + \varepsilon_0 p_n, & \text{if } \langle p_n, c\rangle \leq 0 \\ u_* - \varepsilon_0 p_n, & \text{if } \langle p_n, c\rangle > 0 \end{cases}.$$

Hence, $v_n \in U$ and sequence (v_n) is minimizing. Since $U_* = \{u_* + Ker\, A) \cap U$, it follows that for every $v_* \in U_*$ there exists $x(v_*) \in Ker\, A$ such that $v_* = u_* + x(v_*)$. From

$$\|v_n - v_*\|^2 = \|u_* \pm \varepsilon_0 p_n - u_* - x(v_*)\|^6 = \varepsilon_0^2 + \|x(v_*)\|^2 \geq \varepsilon_0^2$$

it follows that the sequence $(d(u_n, U_*))$ does not converge to 0 as $n \to \infty$. This completes the proof of Theorem.

Let us note that the conditions of the Theorem 3.2 do not guarantee the well-posedness, because they do not eliminate the conditions of the previous theorem.

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