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CONDITIONS FOR UNIQUE RECONSTRUCTION OF SPARSE SIGNALS USING COMPRESSIVE SENSING METHODS

Abstract

A signal is sparse in one of its representation domain if the number of nonzero coefficients in that domain is much smaller than the total number of coefficients. Sparse signals can be reconstructed from a very reduced set of measurements/observations. The topic of this paper are conditions for the unique reconstruction of sparse signals from reduced set of observations. After the basic definitions are introduced, the unique reconstruction conditions are reviewed using the spark, restricted isometry, and coherence of the measurement matrix. Uniqueness of the reconstruction of signals sparse in the discrete Fourier domain (DFT), as the most important signal transformation domain, is considered as well.

USLOVI ZA JEDINSTVENU REKONSTRUKCIJU RIJETKIH SIGNALA METODAMA KOMPRESIVNOG ODABIRANJA

Sažetak

Signal je rijedak u nekom transformacionom domenu ukoliko je broj nenultih koeficijenata signala u tom domenu mnogo manji od ukupnog broja koeficijenata. Rijetki signali mogu biti rekonstruisani na osnovu redukovanog skupa mjerenja (opservacija). Tema ovog rada su uslovi pod kojima je rekonstrukcija rijetkog signala iz redukovanog skupa mjerenja jedinstvena. Nakon definisanja osnovnih pojmova, dati su uslovi

jedinstvenosti rekonstrukcije zasnovani na spark-u, ograničenoj izometriji i koherentnosti mjerne matrice. Na kraju je razmatrana jedinstvenost rekonstrukcije signala koji su rijetki u domenu diskretne Fourierove transformacije, najčešće korišćene transformacije u obradi diskretnih signala.

1 Introduction

A discrete-time signal can be transformed into other domains using different signal transformations. Some signals that cover the whole considered interval in one domain could be sparse in a transformation domain, i.e., could be located within a few nonzero coefficients. An observation or measurement is a linear combination of sparsity domain coefficients. Since the signal samples are linear combinations of the signal transformation coefficients they could be considered as the observations of a sparse signal in the transformation domain. Compressive sensing is a field dealing with a model for data acquisition including the problem of sparse signal recovery from a reduced set of observations [1–13]. A reduced set of observations can be a result of a desire to sense a sparse signal with the lowest possible number of measurements/observations (compressive sensing). It can also be a result of a physical or measurement unavailability to take a complete set of observations [3]. In applications it could happen that some arbitrarily positioned samples of a signal are so heavily corrupted by disturbances that it is better to omit them and consider as unavailable in the analysis and to try to reconstruct the signal with a reduced set of samples [14–16]. Although the reduced set of observations/samples appears in the first case as a result of user strategy to compress the information, while in the next two cases the reduced set of samples is not a result of user intention, all of them can be considered within the unified framework. Under some conditions, a full reconstruction of a sparse signal can be performed with a reduced set of observations/samples, as in the case if a complete set of samples/observations were available [4, 17–20]. A priori information about the nature of the analyzed signal, i.e., its sparsity in a known transformation domain, must be used in this analysis. Sparsity is the main requirement that should be satisfied in order to efficiently apply the compressive sensing methods for sparse signal reconstruction.

Compressive sensing methods are successfully applied to many fields, including radar signal processing [21–26], time-frequency analysis [21, 27–29], L-statistics [16, 30], data hiding [31], communications [32], image processing [33, 34], etc.

Topic of this paper are conditions for the unique reconstruction of sparse signals from reduced set of observations/samples. The basic idea for unique reconstruction will be introduced through an illustrative and simple example in the next section. Then the unique reconstruction condition will be explained within the spark, restricted isometry, and coherence framework. A special case of the signals sparse in the discrete Fourier domain (DFT), as the most important signal transformation domain, will be considered at the end. A simple uniqueness criterion will be presented and illustrated on an example.

2 Illustrative Examples

Consider a large set of N numbers $X(0), X(1), \dots, X(N-1)$. Assume that only one of them is nonzero. We do not know either its position or its value. The aim is to find the position and the value of this number. This case can be related to many real life examples when we have to find one sample which differs from other $N-1$ samples. The problem can easily be reformulated to the case when only one number differs from the expected and known value, and all other assume their expected-known values.

The nonzero value at an position i will be denoted by $X(i)$. A direct way to find the position i of nonzero sample would be to perform up to N measurements and compare each $X(m)$ with zero. However, if N is very large and there is only one nonzero sample we can get the result with just a few observations/measurements. A procedure for the reduced number of observations/measurements is described next.

Take random numbers as weighting coefficients a_i , $i = 0, 1, 2, \dots, N-1$, for each coefficient. Measure the total value of all N weighted coefficients, with weights a_i . Since only one of them is different from the known expected values m_i (or from zero) we will get the total measured value

$$G = a_1 m_1 + a_2 m_2 + \dots + a_i (m_i + X(i)) + \dots + a_N m_N.$$

Next we will subtract the expected value $G_T = a_1 m_1 + a_2 m_2 + \dots + a_N m_N$ from G . The obtained observation/measurement, denoted by $y(0)$, is

$$y(0) = G - G_T = \sum_{k=0}^{N-1} a_k X(k) = a_i X(i),$$

since the nonzero value in the space of $X(0), X(1), \dots, X(N-1)$ is at one position only, $X(k) = X(i)\delta(k-i)$, $k = 0, 1, \dots, N-1$.

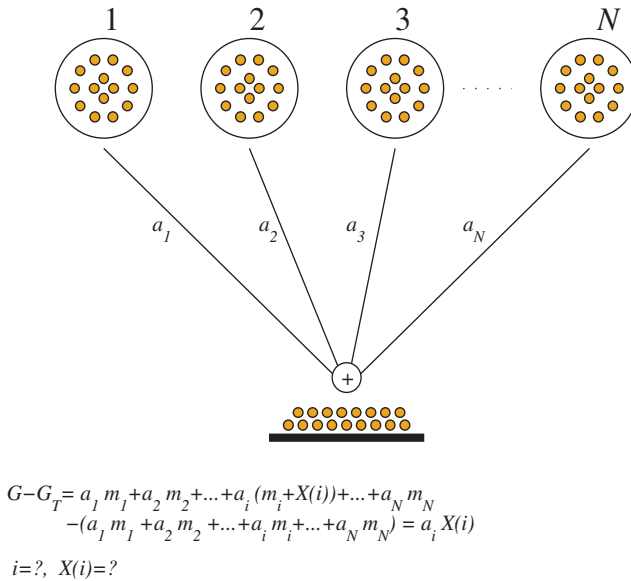


Figure 1: There are N bags with coins. One of them, at an unknown position, contains false coins. False coins differ from the true ones in mass for unknown $X(i) = \Delta m$. The mass of the true coins in the i th bag is m_i .

As an illustration consider a set of N bags with coins. Assume that only one bag contains false coins of a weight $m_i + X(i)$. It is different from the known weights m_i of true coins in bag i . The goal is to find the position and the difference in weight of false coins. From each of N bags we will take a_i , $i = 1, 2, \dots, N$, coins, respectively. Number of coins taken from the i th bag is denoted by a_i . The total measured weight of all coins from N bags is M , Fig.1.

After the expected value is subtracted the observation/measurement $y(0)$ is obtained

$$y(0) = \sum_{k=0}^{N-1} X(k) \psi_k(0), \quad (1)$$

where the weighting coefficients for this measurement are denoted by $\psi_k(0) = a_k$, $k = 0, 1, \dots, N - 1$. In the space of unknowns (variables) $X(0), X(1), \dots, X(N - 1)$ this equation represents an N -dimensional hyperplane. We know that only one unknown $X(k)$ is nonzero at an unknown position $k = i$. The intersection of hyperplane (1) with any of the coordinate axes could be a solution of our problem.

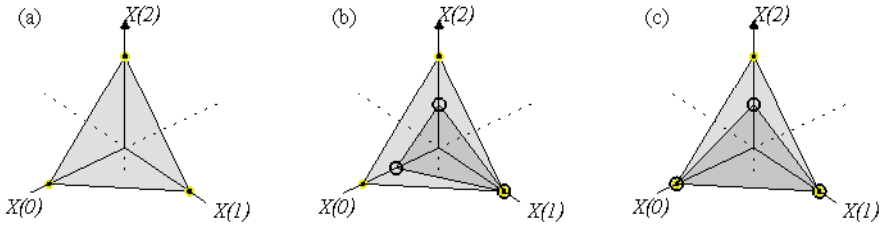


Figure 2: The solution illustration for $N = 3$, $K = 1$, and various possible cases: (a) Three possible solutions for one measurement plane. (b) Unique solution for two measurement planes. (c) Two possible solutions for two measurement planes.

Assuming that a single $X(k)$ is nonzero, a solution will exist for any k . Thus, one measurement would produce a set of N possible single nonzero values equal to

$$X(k) = y(0) / \psi_k(0), \quad \psi_k(0) \neq 0, k = 0, 1, 2, \dots, N - 1.$$

As expected, from one measurement we are not able to solve the problem and to find the position and the value of nonzero sample.

For $N = 3$ possible solutions are illustrated with circles in Fig.2a), denoting intersections of measurements hyperplane with coordinate axes.

If we perform one more measurement $y(1)$, with another set of weighting coefficients $\psi_k(1)$, $k = 0, 1, \dots, N - 1$, and get measured value $y(1) = X(i)\psi_i(1)$ the result will be a hyperplane

$$y(1) = \sum_{k=0}^{N-1} X(k)\psi_k(1).$$

This measurement will produce a new set of possible solutions for each $X(k)$ defined by

$$X(k) = y(1) / \psi_k(1), \quad k = 0, 1, 2, \dots, N - 1.$$

If these two hyperplanes (sets of solutions) produce only one common value

$$X(i) = y(0) / \psi_i(0) = y(1) / \psi_i(1).$$

In a matrix form these two measurements can be written as

$$\begin{bmatrix} y(0) \\ y(1) \end{bmatrix} = \begin{bmatrix} \psi_0(0) & \psi_1(0) & \dots & \psi_{N-1}(0) \\ \psi_0(1) & \psi_1(1) & \dots & \psi_{N-1}(1) \end{bmatrix} \begin{bmatrix} X(0) \\ X(1) \\ \dots \\ X(N-1) \end{bmatrix}$$

$$\mathbf{y} = \mathbf{A}\mathbf{X}$$

where \mathbf{A} is the matrix of coefficients (measurement matrix)

$$\mathbf{A} = \begin{bmatrix} \psi_0(0) & \psi_1(0) & \dots & \psi_{N-1}(0) \\ \psi_0(1) & \psi_1(1) & \dots & \psi_{N-1}(1) \end{bmatrix}$$

and \mathbf{y} are observations/measurements of sparse variable \mathbf{X} .

Common value for two measurements $X(i) = y(0)/\psi_i(0)$ and $X(i) = y(1)/\psi_i(1)$ is unique if

$$\psi_i(0)\psi_k(1) - \psi_i(1)\psi_k(0) \neq 0$$

for any $i \neq k$.

In order to prove this statement assume that two different solutions $X(i)$ and $X(k)$, for the case of one nonzero coefficient, satisfy the same measurement hyperplane equations

$$\psi_i(0)X(i) = y(0), \quad \psi_i(1)X(i) = y(1)$$

and

$$\psi_k(0)X(k) = y(0), \quad \psi_k(1)X(k) = y(1).$$

Then

$$\psi_i(0)X(i) = \psi_k(0)X(k)$$

and

$$\psi_i(1)X(i) = \psi_k(1)X(k).$$

If we divide these two equations we get

$$\psi_i(0)/\psi_i(1) = \psi_k(0)/\psi_k(1)$$

or $\psi_i(0)\psi_k(1) - \psi_i(1)\psi_k(0) = 0$. This is contrary to the assumption that $\psi_i(0)\psi_k(1) - \psi_i(1)\psi_k(0) \neq 0$.

The same conclusion can be made considering matrix form relations for $X(i)$ and $X(k)$. If both of them may satisfy the same two measurements then

$$\begin{aligned} \begin{bmatrix} y(0) \\ y(1) \end{bmatrix} &= \begin{bmatrix} \psi_i(0) & \psi_k(0) \\ \psi_i(1) & \psi_k(1) \end{bmatrix} \begin{bmatrix} X(i) \\ 0 \end{bmatrix} \\ \begin{bmatrix} y(0) \\ y(1) \end{bmatrix} &= \begin{bmatrix} \psi_i(0) & \psi_k(0) \\ \psi_i(1) & \psi_k(1) \end{bmatrix} \begin{bmatrix} 0 \\ X(k) \end{bmatrix}. \end{aligned} \quad (2)$$

Subtraction of the previous matrix equations results in

$$\begin{bmatrix} \psi_i(0) & \psi_k(0) \\ \psi_i(1) & \psi_k(1) \end{bmatrix} \begin{bmatrix} X(i) \\ -X(k) \end{bmatrix} = 0.$$

For $\psi_i(0)\psi_k(1) - \psi_i(1)\psi_k(0) \neq 0$ follows $X(i) = X(k) = 0$. Therefore two different nonzero solutions $X(i)$ and $X(k)$ in this case cannot exist. This concludes the proof that the solution is unique if

$$\psi_i(0)\psi_k(1) - \psi_i(1)\psi_k(0) = \det \begin{bmatrix} \psi_i(0) & \psi_k(0) \\ \psi_i(1) & \psi_k(1) \end{bmatrix} \neq 0$$

for any $i \neq k$. It also means that $\text{rank}(\mathbf{A}_2) = 2$ for any \mathbf{A}_2 being a 2×2 submatrix of the matrix of coefficients (measurement matrix) \mathbf{A} .

Let us consider M measurements in this example. Since we have assumed that only one coefficient $X(i)$ is nonzero it will satisfy all measurements

$$\psi_i(0)X(i) = y(0), \quad \psi_i(1)X(i) = y(1), \dots, \quad \psi_i(M-1)X(i) = y(M-1).$$

The solution will not be unique if there is another coefficient $X(k)$, $k \neq i$ satisfying

$$\psi_k(0)X(k) = y(0), \quad \psi_k(1)X(k) = y(1), \dots, \quad \psi_k(M-1)X(k) = y(M-1).$$

Then the corresponding coefficients of the measurement matrix satisfy

$$\frac{\psi_i(0)}{\psi_k(0)} = \frac{\psi_i(1)}{\psi_k(1)} = \dots = \frac{\psi_i(M-1)}{\psi_k(M-1)}.$$

In this case measurement matrix is

$$\mathbf{A} = \begin{bmatrix} \psi_0(0) & \psi_1(0) & \dots & \psi_{N-1}(0) \\ \psi_0(1) & \psi_1(1) & \dots & \psi_{N-1}(1) \\ \vdots & \vdots & \ddots & \vdots \\ \psi_0(M-1) & \psi_1(M-1) & \dots & \psi_{N-1}(M-1) \end{bmatrix}$$

The solution is not unique if any two columns are linearly dependent. The uniqueness requires that all two column submatrices \mathbf{A}_2 of \mathbf{A} are of rank 2. The determinant for all $\mathbf{A}_2^T \mathbf{A}_2$ is nonzero.

In numerical and practical applications we would not be satisfied, if for example $\det(\mathbf{A}_2^T \mathbf{A}_2) \neq 0$ but $\det(\mathbf{A}_2^T \mathbf{A}_2) = \varepsilon$ close to zero. In this case the theoretical condition for a unique solution would be satisfied, however the analysis and possible inversion would be highly sensitive to any kind of noise, including quantization noise. Thus, a practical requirement is that the determinant is not just different from zero, but that it sufficiently differs from zero so that an inversion stability and robustness to a noise is achieved. Inversion stability for a matrix $\mathbf{B} = \mathbf{A}_2^T \mathbf{A}_2$ is commonly described by the condition number of matrix

$$\text{cond} \{\mathbf{B}\} = \frac{\lambda_{\max}}{\lambda_{\min}}$$

where λ_{\max} and λ_{\min} are the largest and the smallest eigenvalue of matrix \mathbf{B} . The inversion stability worsens as λ_{\min} approaches to zero (when λ_{\min} is small as compared to λ_{\max}). For stable and robust calculations a requirement

$$\frac{\lambda_{\max}}{\lambda_{\min}} \leq 1 + \delta$$

is imposed, with a nonnegative constant δ being sufficiently small. In our example this condition should hold for all submatrices \mathbf{A}_2 .

As a next example consider a signal described by a weighted sum of K harmonics from a set of possible oscillatory functions $e^{j2\pi kn/N}$, $k = 0, 1, 2, \dots, N-1$,

$$x(n) = A_1 e^{j2\pi k_1 n/N} + A_2 e^{j2\pi k_2 n/N} + \dots + A_K e^{j2\pi k_K n/N},$$

with $K \ll N$. In the DFT domain this signal will be sparse with $X(k) = \text{DFT}\{x(n)\}$ having only few nonzero values at $k = k_i$, $i = 1, 2, \dots, K$. According to the sampling theorem the sampling of this kind of signals should be adjusted to the maximal expected signal frequency $k = \max\{k_1, k_2, \dots, k_K\}$. For an arbitrary set of frequencies, it means that we should adjust sampling to the maximal possible frequency $k = N-1$ and to use the full set of N signal values/measurements at $n = 0, 1, 2, \dots, N-1$ in order to avoid aliasing.

However, if we know that the signal consists of only $K \ll N$ functions with unknown amplitudes, then regardless of their frequencies, the signal can be fully reconstructed from a reduced set of samples. Samples can be considered as weighted measurements of the sparse function $X(k)$,

$$y(0) = x(n_1) = \sum_{k=0}^{N-1} X(k) \psi_k(n_1),$$

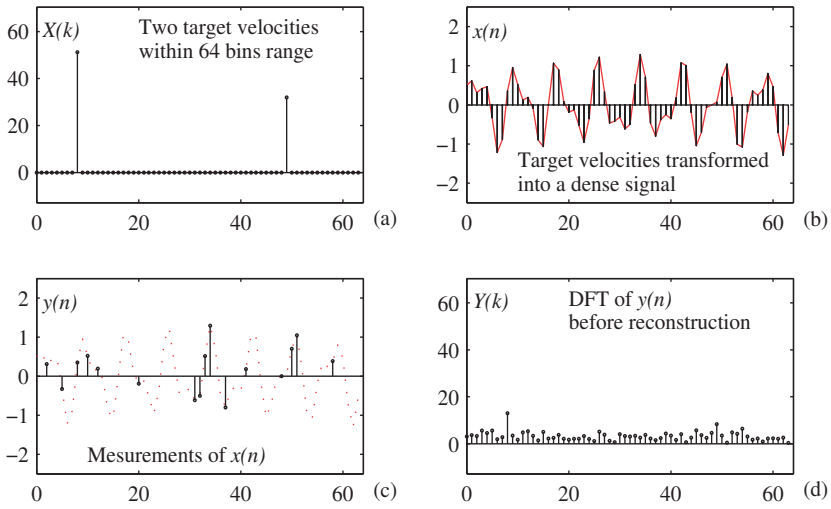


Figure 3: (a) Signal in the frequency domain, where it is sparse (for example, velocities of two targets in Doppler radar signal). (b) Signal in the time domain, where it is dense. (c) Reduced set of measurements (samples) and (d) its DFT before reconstruction, calculated using the available samples only. Real parts of signals are presented.

with the weighting coefficients $\psi_k(n_1) = \exp(j2\pi n_1 k/N)/N$. The previous relation is the IDFT. Now a similar analysis like in the previous illustrative example can be performed, assuming for example $K = 1$ or $K = 2$. We can find the position and the value of nonzero $X(k)$ using just a few signal samples $y(i)$.

This model corresponds to many signals in real life. For example, in the Doppler-radar systems the speed of a radar target is transformed into a frequency of a sinusoidal signal [23,24]. Since the returned signal contains only one or just a few targets, the signal representing target velocity is a sparse signal in the DFT domain. It can be reconstructed from fewer samples than the total number of radar return signal samples N , Fig.3.

In signal processing the DFT as the domain of signal sparsity is commonly used, since it plays the central role in engineering applications [3,20]. Note that in the compressive sensing theory random measurement matrices are mainly used. The compressive sensing results and algorithms are used as a tool to solve the problems involving sparse signals.

3 Definitions

A big set of discrete-time data $x(n), n = 0, 1, \dots, N - 1$, with a large number of samples N is considered. Its coefficients in a transformation domain are denoted as

$$\mathbf{X} = [X(0), X(1), \dots, X(N - 1)]^T,$$

where T represents the transpose operation. We consider a signal to be sparse in this transformation domain if the number of nonzero coefficients K is much smaller than the number of the original signal samples N , i.e., if $X(k) = 0$ for $k \notin \mathbb{K} = \{k_1, k_2, \dots, k_K\}$ and $K \ll N$. The number of nonzero coefficients is commonly denoted by $\|\mathbf{X}\|_0$

$$\|\mathbf{X}\|_0 = \text{card} \{\mathbb{K}\} = K,$$

where $\text{card} \{\mathbb{K}\}$ is the cardinality of set \mathbb{K} . It is equal to the number of elements in \mathbb{K} . It is called the ℓ_0 -norm (norm-zero) or the ℓ_0 -pseudo-norm of vector \mathbf{X} although it does not satisfy the norm properties.

The observations/measurements are defined as linear combinations of signal coefficients in the sparsity domain

$$y(m) = \sum_{k=0}^{N-1} X(k)\psi_k(m), \quad (3)$$

where $m = 0, 1, \dots, M - 1$ is the measurement index and $\psi_k(m)$ are the weighting coefficients. The vector form of the measurement signal is denoted by \mathbf{y}

$$\mathbf{y} = [y(0), y(1), \dots, y(M - 1)]^T.$$

The measurements defined by Eq.(3) can be written as a undetermined system of $M < N$ equations

$$\begin{bmatrix} y(0) \\ y(1) \\ \vdots \\ y(M-1) \end{bmatrix} = \begin{bmatrix} \psi_0(0) & \psi_1(0) & \cdots & \psi_{N-1}(0) \\ \psi_0(1) & \psi_1(1) & \cdots & \psi_{N-1}(1) \\ \vdots & \vdots & \ddots & \vdots \\ \psi_0(M-1) & \psi_1(M-1) & \cdots & \psi_{N-1}(M-1) \end{bmatrix} \begin{bmatrix} X(0) \\ X(1) \\ \vdots \\ X(N-1) \end{bmatrix}$$

or using matrix notation

$$\mathbf{y} = \mathbf{A}\mathbf{X}$$

where \mathbf{A} is the measurement matrix of size $M \times N$.

The fact that the signal is sparse with $X(k) = 0$ for $k \notin \mathbb{K} = \{k_1, k_2, \dots, k_K\}$ is not included in the measurement matrix \mathbf{A} since the positions of the nonzero

values are unknown. If the knowledge that $X(k) = 0$ for $k \notin \mathbb{K}$ were included then a reduced system would be obtained as

$$\begin{bmatrix} y(0) \\ y(1) \\ \vdots \\ y(M-1) \end{bmatrix} = \begin{bmatrix} \psi_{k_1}(0) & \psi_{k_2}(0) & \cdots & \psi_{k_K}(0) \\ \psi_{k_1}(1) & \psi_{k_2}(1) & \cdots & \psi_{k_K}(1) \\ \vdots & \vdots & \ddots & \vdots \\ \psi_{k_1}(M-1) & \psi_{k_2}(M-1) & \cdots & \psi_{k_K}(M-1) \end{bmatrix} \begin{bmatrix} X(k_1) \\ X(k_2) \\ \vdots \\ X(k_K) \end{bmatrix}$$

with a reduced $M \times K$ measurement matrix \mathbf{A}_K defined as

$$\mathbf{y} = \mathbf{A}_K \mathbf{X}_K. \quad (4)$$

This is an overdetermined system of equation, $K < N$. Matrix \mathbf{A}_K would be formed if we assumed/knew the positions of nonzero samples $k \in \mathbb{K}$. It would follow from the measurement matrix \mathbf{A} by omitting the columns corresponding to the zero-valued coefficients in \mathbf{X} .

3.1 Common Measurement Matrices

Some common measurement matrices used in practical applications and theoretical considerations will be presented here.

Randomness of measurement matrices is a favorable property in compressive sensing and matrices with random elements are often used. The most common is the measurement matrix with zero-mean unity variance Gaussian distributed numbers as elements

$$\phi_k(n) \sim \frac{1}{\sqrt{M}} \mathcal{N}(0, 1)$$

normalized with $1/\sqrt{M}$ so that the energy of each column is one.

In signal processing the most common transform is the DFT. The coefficients of its direct transform matrix Φ are defined as

$$\phi_k(n) = \exp(-j2\pi nk/N).$$

The inverse DFT matrix coefficients are $\psi_k(n) = \frac{1}{N} \exp(j2\pi nk/N)$. Commonly the measurements are the signal samples $y(m-1) = x(n_m)$ for $m = 1, \dots, M$ where

$$n_m \in \mathbb{M} = \{n_1, n_2, \dots, n_M\} \subset \{0, 1, \dots, N-1\},$$

and

$$y(m-1) = x(n_m) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) e^{j2\pi n_m k/N}.$$

Therefore, the measurement matrix is obtained by keeping the rows of the inverse DFT matrix corresponding to the samples at $n_m \in \{0, 1, \dots, N - 1\}$, for the measurements $m = 1, 2, \dots, M$,

$$\mathbf{A} = \frac{1}{N} \begin{bmatrix} 1 & e^{j2\pi n_1/N} & \dots & e^{j2\pi n_1(N-1)/N} \\ 1 & e^{j2\pi n_2/N} & \dots & e^{j2\pi n_2(N-1)/N} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & e^{j2\pi n_M/N} & \dots & e^{j2\pi n_M(N-1)/N} \end{bmatrix}. \quad (5)$$

This is a partial inverse DFT matrix. In compressive sensing theory it is common to normalize the measurement matrix so that the energy of its columns (diagonal elements of $\mathbf{A}^H \mathbf{A}$ matrix) is equal to one. Then the factor $1/N$ in \mathbf{A} should be replaced by $1/\sqrt{M}$.

In order to increase randomness in the Fourier transform matrix, the measurements may be taken at any random instant. Then the measurement vector elements are $y(m-1) = x(t_m)$ where t_m , $m = 1, 2, \dots, M$ are random instants within the considered time interval T . The measurement matrix follows then from the Fourier series definition $x(t) = \sum_{k=0}^{N-1} X(k) \exp(j2\pi kt/T)$. It has been assumed that the Fourier series coefficients are within $0 \leq k \leq N-1$. The measurements matrix is

$$\mathbf{A} = \begin{bmatrix} 1 & e^{j2\pi t_1/T} & \dots & e^{j2\pi t_1(N-1)/T} \\ 1 & e^{j2\pi t_2/T} & \dots & e^{j2\pi t_2(N-1)/T} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & e^{j2\pi t_M/T} & \dots & e^{j2\pi t_M(N-1)/T} \end{bmatrix} \quad (6)$$

with a possible normalization factor $1/\sqrt{M}$. This measurement matrix is a partial random inverse Fourier transform matrix.

4 Reconstruction Problem Formulation

The signal can be reconstructed from its measurements defined by vector \mathbf{y} by finding the sparsest vector \mathbf{X} that corresponds to the measurements \mathbf{y} . Hence, by introducing the notation for the number of components based on the ℓ_0 -norm $K = \|\mathbf{X}\|_0$, the fundamental minimization problem can be formulated as:

$$\min \|\mathbf{X}\|_0 \text{ subject to } \mathbf{y} = \mathbf{A}\mathbf{X}. \quad (7)$$

In general, the ℓ_0 -norm is not very suitable for most minimization methods. However, a class of algorithms is based on the minimization of the num-

ber of coefficients $K = \|\mathbf{X}\|_0$ in an implicit way. For instance, in certain applications we can predict the number of components or we are able to estimate the position of non-zero coefficients. Thus, compared to the direct search method, the computational complexity will be significantly reduced if we are able to estimate positions of nonzero coefficients and solve the problem with the minimal possible number of nonzero coefficients. The important class of this algorithms are *matching pursuit* (MP) algorithms.

Minimization of the number of non-zero coefficients using the ℓ_0 -norm is a nonconvex optimization problem that cannot be solved using well developed iterative algorithms and linear programming methods [35–42]. To avoid dealing with NP-hard problems, significant efforts have been undertaken to replace the nonconvex and discontinuous ℓ_0 -norm with a convex and continuous norm that would be more appropriate for optimization. As a result, the ℓ_1 -norm (norm-one) has been commonly employed in many signal reconstruction approaches [43–46]. It has been shown that, under certain conditions, minimization of the ℓ_1 -norm produces the same solution as the minimization of the ℓ_0 -norm.

In the ℓ_1 -norm based reconstructions the problem is formulated as

$$\min \|\mathbf{X}\|_1 \quad \text{subject to} \quad \mathbf{y} = \mathbf{A}\mathbf{X}$$

where

$$\|\mathbf{X}\|_1 = \sum_{k=0}^{N-1} |X(k)|.$$

This is the so-called *basis pursuit* (BP) approach to sparse signal reconstruction.

4.1 Conditions for Unique Reconstruction

4.1.1 Spark

The spark of a matrix \mathbf{A} is defined as the smallest number of linearly dependent columns of \mathbf{A} . In other words if $\text{spark}\{\mathbf{A}\} = K$, then any collection of $K_1 < K$ columns of \mathbf{A} are linearly independent.

Spark can also be defined as a minimal number of nonzero entries in a vector $\mathbf{X} \neq \mathbf{0}$ such that $\mathbf{A}\mathbf{X} = \mathbf{0}$

$$\min \|\mathbf{X}\|_0 \quad \text{such that} \quad \mathbf{A}\mathbf{X} = \mathbf{0}$$

If matrix \mathbf{A} is of size $N \times K$ with $N > K$ and $\text{spark}\{\mathbf{A}\} = K + 1$ then all $K \times K$ submatrices of matrix \mathbf{A} are nonsingular, i.e., with nonzero determinant.

The analysis of a signal with an arbitrary sparsity K is similar to the analysis for $K = 1$. To get the first set of possible solutions for K nonzero coefficients (of sparsity K) we need K measurements. For any combination of K (out of N) nonzero coefficients $X(k)$, $k \in \{k_1, k_2, \dots, k_K\}$, we will get a possible solution. There exist $\binom{N}{K}$ such possible combinations/solutions. Additional K measurements will be used to produce another set of $\binom{N}{K}$ possible solutions. The intersection of these two sets is then the solution of our problem.

Consider the case when the number of measurements M is twice higher than the sparsity K , $M = 2K$. The K -sparse solution is unique if the determinants of all \mathbf{A}_{2K} submatrices of matrix \mathbf{A} are different from zero.

This statement will be proven by contradiction. Assume that $M = 2K$ measurements are available within the vector \mathbf{y} . Assume that two different solutions for \mathbf{X} of sparsity K exist. Denote the nonzero parts of the solutions by $\mathbf{X}_K^{(1)}$ and $\mathbf{X}_K^{(2)}$. Both of them satisfy the measurements equation,

$$\mathbf{A}_K^{(1)} \mathbf{X}_K^{(1)} = \mathbf{y}$$

and

$$\mathbf{A}_K^{(2)} \mathbf{X}_K^{(2)} = \mathbf{y},$$

where $\mathbf{A}_K^{(1)}$ and $\mathbf{A}_K^{(2)}$ are two different submatrices of matrix \mathbf{A} of size $M \times K$ corresponding to the elements in $\mathbf{X}_K^{(1)}$ and $\mathbf{X}_K^{(2)}$. If we rewrite these equations by adding zeros

$$\begin{bmatrix} \mathbf{A}_K^{(1)} & \mathbf{A}_K^{(2)} \end{bmatrix} \begin{bmatrix} \mathbf{X}_K^{(1)} \\ \mathbf{0}_K \end{bmatrix} = \mathbf{y} \quad \text{and} \quad \begin{bmatrix} \mathbf{A}_K^{(1)} & \mathbf{A}_K^{(2)} \end{bmatrix} \begin{bmatrix} \mathbf{0}_K \\ \mathbf{X}_K^{(2)} \end{bmatrix} = \mathbf{y} \quad (8)$$

and subtract them we get

$$\begin{bmatrix} \mathbf{A}_K^{(1)} & \mathbf{A}_K^{(2)} \end{bmatrix} \begin{bmatrix} \mathbf{X}_K^{(1)} \\ -\mathbf{X}_K^{(2)} \end{bmatrix} = \mathbf{0}. \quad (9)$$

There are no nonzero solutions for $\mathbf{X}_K^{(1)}$ and $\mathbf{X}_K^{(2)}$ if the determinant of matrix $\mathbf{A}_{2K} = \begin{bmatrix} \mathbf{A}_K^{(1)} & \mathbf{A}_K^{(2)} \end{bmatrix}$ is nonzero. If all possible submatrices \mathbf{A}_{2K} (including all lower order submatrices) of measurement matrix \mathbf{A} are nonsingular then two solutions of sparsity K cannot exist, and the solution is unique. Note that there are $\binom{N}{2K}$ submatrices \mathbf{A}_{2K} .

Based on the previous analysis, the solution for a K sparse problem is unique if

$$\text{spark}\{\mathbf{A}\} > 2K.$$

For $M > 2K$ the matrix $\mathbf{A}_{2K} = \begin{bmatrix} \mathbf{A}_K^{(1)} & \mathbf{A}_K^{(2)} \end{bmatrix}$ dimension is $M \times 2K$. Again if $\text{rank}(\mathbf{A}_{2K}) = \text{rank}(\mathbf{A}_{2K}^T \mathbf{A}_{2K}) = 2K$ system (9) does not have a nonzero solution. It means that the reconstruction is unique. If $\text{rank}(\mathbf{A}_{2K}) = \text{rank}(\mathbf{A}_{2K}^T \mathbf{A}_{2K}) = 2K$ for all submatrices \mathbf{A}_{2K} then $\text{spark}\{\mathbf{A}\} > 2K$.

If the vector \mathbf{X} is of sparsity K , with $\|\mathbf{X}\|_0 = K$ then if

$$K < \frac{1}{2} \text{spark}\{\mathbf{A}\}$$

the solution \mathbf{X} is unique.

In order to prove this statement (that has been already explained) consider a measurement matrix \mathbf{A} whose spark is $\text{spark}\{\mathbf{A}\}$. Then for a sparse vector \mathbf{X} of sparsity $K = \text{spark}\{\mathbf{A}\}$ obviously there exists such a combination of nonzero elements in \mathbf{X} so that they coincide with the dependent columns. Then we can obtain

$$\mathbf{A}\mathbf{X} = \mathbf{0}.$$

This property is used for the spark definition as well.

Note that for any \mathbf{X} of sparsity $K < \text{spark}\{\mathbf{A}\}$ the relation $\mathbf{A}\mathbf{X} = \mathbf{0}$ will not hold, since corresponding independent columns of \mathbf{A} multiplied with nonzero elements of \mathbf{X} cannot produce a zero result. Since $K < \text{spark}\{\mathbf{A}\}$ it means that all sets of K columns from \mathbf{A} are independent.

The proof of the previous statement will be based on the contradiction. Assume that \mathbf{X} is a solution of $\mathbf{A}\mathbf{X} = \mathbf{y}$ and that its sparsity satisfies $K < \frac{1}{2} \text{spark}\{\mathbf{A}\}$. Assume also that there is another solution \mathbf{H} such that $\mathbf{A}\mathbf{H} = \mathbf{y}$ and \mathbf{H} is also sparse with sparsity lower than $\frac{1}{2} \text{spark}\{\mathbf{A}\}$. Since

$$\mathbf{A}\mathbf{H} = \mathbf{A}\mathbf{X} = \mathbf{y}$$

$$\mathbf{A}(\mathbf{H} - \mathbf{X}) = \mathbf{0}$$

then

$$\text{spark}\{\mathbf{A}\} < \|\mathbf{H} - \mathbf{X}\|_0$$

or

$$\text{spark}\{\mathbf{A}\} < \|\mathbf{H} - \mathbf{X}\|_0 \leq \|\mathbf{H}\|_0 + \|\mathbf{X}\|_0$$

$$\text{spark}\{\mathbf{A}\} - \|\mathbf{H}\|_0 \leq \|\mathbf{X}\|_0.$$

The inequality follows from the fact that two nonzero elements, at the same position in \mathbf{H} and \mathbf{X} , can produce a zero element in $\mathbf{H} - \mathbf{X}$, while two zero elements in these vectors cannot produce a nonzero element in $\mathbf{H} - \mathbf{X}$. If there is another solution \mathbf{H} such that $\|\mathbf{H}\|_0 < \frac{1}{2} \text{spark}\{\mathbf{A}\}$ then from the last inequality follows that $\|\mathbf{X}\|_0 > \frac{1}{2} \text{spark}\{\mathbf{A}\}$. This is a contradiction to the assumption that both solutions \mathbf{H} and \mathbf{X} have sparsity lower than $\frac{1}{2} \text{spark}\{\mathbf{A}\}$.

4.1.2 Restricted Isometry Property

Note that for any square matrix its determinant is equal to the product of its eigenvalues

$$\det\{\mathbf{A}_{2K}^T \mathbf{A}_{2K}\} = \lambda_1 \lambda_2 \cdots \lambda_{2K}.$$

The condition that the solution is unique if the determinant of $\mathbf{A}_{2K}^T \mathbf{A}_{2K}$ for all \mathbf{A}_{2K} submatrices of matrix \mathbf{A} are different from zero can be rewritten as

$$\min_i |\lambda_i| > 0.$$

In numerical and practical applications we would not be satisfied, if any of the determinants is very close to zero. In this case the theoretical condition for a unique solution would be satisfied, however the analysis and possible inversion would be highly sensitive to any kind of noise in measurements. Thus, a practical requirement is that the determinant is not just different from zero, but that it sufficiently differs from zero so that an inversion stability and noise robustness is achieved.

From the matrix theory it is known that the norm of a matrix \mathbf{A}_{2K} satisfies

$$\lambda_{\min} \leq \frac{\|\mathbf{A}_{2K} \mathbf{X}_{2K}\|_2^2}{\|\mathbf{X}_{2K}\|_2^2} = \frac{\mathbf{X}_{2K}^T \mathbf{A}_{2K}^T \mathbf{A}_{2K} \mathbf{X}_{2K}}{\mathbf{X}_{2K}^T \mathbf{X}_{2K}} \leq \lambda_{\max}, \quad (10)$$

where λ_{\min} and λ_{\max} are the minimal and the maximal eigenvalue of the matrix $\mathbf{A}_{2K}^T \mathbf{A}_{2K}$ and $\|\mathbf{X}\|_2^2 = |X(0)|^2 + \dots + |X(N-1)|^2$ is the squared ℓ_2 -norm (norm-two).

The isometry property for a linear transformation matrix \mathbf{A} holds if

$$\|\mathbf{A}\mathbf{X}\|_2^2 = \|\mathbf{X}\|_2^2 \quad \text{or} \quad \frac{\|\mathbf{A}\mathbf{X}\|_2^2}{\|\mathbf{X}\|_2^2} = 1.$$

The restricted isometry property (RIP) for a matrix \mathbf{A}_{2K} and a $2K$ -sparse vector \mathbf{X}_{2K} holds if

$$1 - \delta_{2K} \leq \frac{\|\mathbf{A}_{2K} \mathbf{X}_{2K}\|_2^2}{\|\mathbf{X}_{2K}\|_2^2} \leq 1 + \delta_{2K}, \quad (11)$$

where $0 \leq \delta_{2K} < 1$ is the isometric constant. From Eqs.(10) and (11) we can write

$$\delta_{2K} = \max\{1 - \lambda_{\min}, \lambda_{\max} - 1\}.$$

Commonly, isometric constant is defined by $\lambda_{\max} - 1$ and it is calculated as maximal eigenvalue of matrix $\mathbf{A}_{2K}^T \mathbf{A}_{2K} - \mathbf{I}$. Normalized energies of the

columns of matrix \mathbf{A} (diagonal elements of $\mathbf{A}_{2K}^T \mathbf{A}_{2K}$) are assumed. Otherwise, the normalization factors should be added. For complex-valued matrices Hermitian transpose should be used in $\mathbf{A}_{2K}^H \mathbf{A}_{2K}$.

For a K -sparse vector \mathbf{X} and a measurement matrix \mathbf{A} the RIP is satisfied if relation (11) holds for all submatrices \mathbf{A}_K with $0 \leq \delta_K < 1$. The solution for K -sparse vector is unique if the measurement matrix satisfy the RIP for $2K$ -sparse vector \mathbf{X} with $0 \leq \delta_{2K} < 1$.

Note that if the RIP is satisfied then $\lambda_{\min} > 0$. It means that there is no \mathbf{A}_{2K} submatrix of \mathbf{A} such that $\mathbf{A}_{2K}^H \mathbf{A}_{2K}$ is a singular matrix. The uniqueness proof reduces to the previous one.

Restricted isometry property for small δ_{2K} is closer to the isometry property and improves the solution stability. It can be related to the matrix conditional number. The conditional number of a matrix $\mathbf{A}_{2K}^T \mathbf{A}_{2K}$ is defined as the ratio of its maximal and minimal eigenvalues

$$\text{cond} \left\{ \mathbf{A}_{2K}^T \mathbf{A}_{2K} \right\} = \frac{\lambda_{\max}}{\lambda_{\min}}.$$

If a matrix \mathbf{A}_{2K} satisfies the restricted isometry property with δ_{2K} then

$$\text{cond} \left\{ \mathbf{A}_{2K}^T \mathbf{A}_{2K} \right\} \leq \frac{1 + \delta_{2K}}{1 - \delta_{2K}}.$$

With small values of δ_{2K} the conditional number is close to one, meaning stable invertibility and low sensitivity to the input noise (small variations of the input signal (measurements) do not cause large variations of the result). Common requirement for this constant is

$$0 \leq \delta_{2K} < \sqrt{2} - 1.$$

The restricted isometry constant within this range will also guarantee the equivalence of the solutions obtained in the reconstruction based on the ℓ_0 -norm and the ℓ_1 -norm minimization [17,18].

4.1.3 Coherence

The mutual coherence (coherence index) of a matrix \mathbf{A} is defined as the maximal absolute value of the normalized scalar product of its two columns

$$\mu = \max |\mu_{mk}|, \text{ for } m \neq k$$

where

$$\mu_{mk} = \frac{\sum_{i=0}^{M-1} \alpha_m(i) \alpha_k^*(i)}{\sqrt{\sum_{i=0}^{M-1} |\alpha_m(i)|^2 \sum_{i=0}^{M-1} |\alpha_k(i)|^2}} \quad (12)$$

and $\alpha_k(i)$ are the elements of the k th column of matrix \mathbf{A} . If $\sum_{i=0}^{M-1} |\alpha_k(i)|^2 = \sum_{i=0}^{M-1} |\alpha_m(i)|^2$ then

$$\mu_{mk} = \frac{\sum_{i=0}^{M-1} \alpha_m(i) \alpha_k^*(i)}{\sum_{i=0}^{M-1} |\alpha_k(i)|^2}. \quad (13)$$

Note that μ_{mk} , $m \neq k$, are the off-diagonal elements of matrix $\mathbf{A}^H \mathbf{A}$ normalized with the corresponding diagonal elements.

This index plays an important role in the analysis of measurement matrices. The coherence index should be as small as possible, or in other words the incoherence is a desirable property. With smaller values of coherence index the matrix $\mathbf{A}^H \mathbf{A}$ is closer to the identity matrix.

The condition that all eigenvalues of matrix $\mathbf{A}_{2K}^H \mathbf{A}_{2K}$ are nonzero can be written in terms of the coherence index. In general, the eigenvalue relation for a matrix $\mathbf{A}_{2K}^H \mathbf{A}_{2K}$ reads

$$(\mathbf{A}_{2K}^H \mathbf{A}_{2K}) \mathbf{u} = \lambda \mathbf{u}$$

where \mathbf{u} denotes an eigenvector. For an eigenvector we can always choose that its maximal coordinate is $u_m = \max_k(u_k) = 1$ and $u_k \leq 1$ for $k \neq m$. Now we can write the general eigenvalue relation in the form

$$\sum_k \mu_{mk} u_k = \lambda u_m = \lambda$$

or

$$\sum_{k, k \neq m} \mu_{mk} u_k = \lambda - \mu_{mm}$$

From this relation we can conclude

$$|\lambda - \mu_{mm}| \leq \sum_{k, k \neq m} |\mu_{mk} u_k| \leq \sum_{k, k \neq m} |\mu_{mk}|.$$

Considering eigenvalue λ as a variable and μ_{mk} as constants we conclude that the last inequality describes a disc area with the center at μ_{mm} and radius $\sum_{k, k \neq m} |\mu_{mk}|$. It does not include point $\lambda = 0$ if

$$\mu_{mm} > \sum_{k, k \neq m} |\mu_{mk}|. \quad (14)$$

Therefore the matrix $\mathbf{A}_{2K}^H \mathbf{A}_{2K}$ will be nonsingular if the above condition is met. This is the Gershgorin circle (disk) theorem.

For normalized matrix $\mathbf{A}_{2K}^H \mathbf{A}_{2K}$ we have $\mu_{mm} = 1$ and

$$\mu = \max_{m \neq k} |\mu_{mk}|.$$

We have already concluded that the solution for K -sparse vector will be unique if for all possible submatrices \mathbf{A}_{2K} the matrices $\mathbf{A}_{2K}^H \mathbf{A}_{2K}$ are nonsingular. Note that the off-diagonal elements of $\mathbf{A}_{2K}^H \mathbf{A}_{2K}$ are a subset of the off-diagonal elements of matrix $\mathbf{A}^H \mathbf{A}$. The same holds for the diagonal elements. It means that the coherence μ of matrix \mathbf{A} will be always greater than or equal to the coherence of any submatrix \mathbf{A}_{2K} .

The nonsingularity condition for all matrices $\mathbf{A}_{2K}^H \mathbf{A}_{2K}$, and unique solution for a K sparse vector \mathbf{X} , is achieved if

$$1 > (2K - 1)\mu$$

or

$$K < \frac{1}{2} \left(1 + \frac{1}{\mu}\right).$$

The proof follows from (14) for normalized matrix $\mathbf{A}^H \mathbf{A}$. The inequality

$$1 = \mu_{mm} > \sum_{k=1, k \neq m}^{2K} |\mu_{mk}|$$

is satisfied if $1 > (2K - 1)\mu$ since $\sum_{k=1, k \neq m}^{2K} |\mu_{mk}| < (2K - 1)\mu$.

The coherence index cannot be arbitrarily small for an $M \times N$ matrix \mathbf{A} ($M < N$). The Welch upper bound relation holds

$$\mu \geq \sqrt{\frac{N - M}{M(N - 1)}}. \quad (15)$$

The Gershgorin circle (disk) theorem can be used to determine the spark lower bound. If the relation $K < (1 + 1/\mu)/2$ holds for a given K then it holds for any order lower than K . It means that all submatrices $\mathbf{A}_{2K}^H \mathbf{A}_{2K}$ of $2K$ and lower order are nonsingular. Therefore the spark of such a matrix is greater than $2K$

$$\text{spark}(\mathbf{A}) > 2K \quad (16)$$

or

$$K < \frac{1}{2} \text{spark}(\mathbf{A})$$

if $K < (1 + 1/\mu)/2$. It means that

$$\text{spark}(\mathbf{A}) \geq 1 + \frac{1}{\mu}.$$

4.2 Numerical Example

Consider 5×8 measurement matrix

$$\mathbf{A} = \begin{bmatrix} 0.1 & 0.1 & 0.3 & -0.7 & 0.7 & -0.1 & 0.1 & 0.3 \\ 0.4 & -0.8 & -0.4 & -0.1 & 0.3 & 0.3 & 0.3 & -0.5 \\ 0.3 & 0.5 & -0.5 & 0.4 & 0.5 & -0.7 & 0.1 & -0.4 \\ -0.7 & -0.3 & 0.1 & 0.3 & 0.4 & -0.5 & 0.5 & -0.7 \\ -0.5 & 0.1 & -0.7 & -0.5 & -0.1 & -0.4 & -0.8 & -0.1 \end{bmatrix}. \quad (17)$$

Columns of this matrix are normalized. The norm of each column is equal to one. The matrix dimensions are small so we can solve NP-hard problems and calculate the spark and the restricted isometry property constants, by checking all possible combinations, in a reasonable time.

4.2.1 Spark Calculation

For the spark calculation we first concluded that there is no all zero column, meaning that $\text{spark}\{\mathbf{A}\} > 1$. Then we have check linear dependence of each pairs of columns. In total $\binom{8}{2}$ combinations are checked by calculating rank of each 2×5 submatrix. In all cases rank was equal to 2 meaning that all pair of two columns are linearly independent. It means that $\text{spark}\{\mathbf{A}\} > 2$. Next all $\binom{8}{3}$ possible combinations of three columns are considered. For all submatrices we concluded that their rank is 3 meaning that there is no a set of three linearly dependent columns in the measurement matrix \mathbf{A} . Therefore $\text{spark}\{\mathbf{A}\} > 3$. Calculation is repeated for all combinations of four and five columns with the same result. The final conclusion is that the spark of this matrix is $\text{spark}\{\mathbf{A}\} = 6$ meaning that all combinations of five and less columns are linearly independent. The uniqueness condition based on matrix spark state that sparsity K limit is

$$K < \frac{1}{2} \text{spark}\{\mathbf{A}\} = 3.$$

According to the spark based uniqueness condition, the reconstruction is unique for $K = 1$ and $K = 2$. We may conclude that if we find a sparse vector \mathbf{X} in the reconstruction with sparsity $K \leq 2$ then this is the sparsest possible solution of our problem.

4.2.2 Coherence Calculation

Coherence of the considered matrix is calculated a maximal absolute value of the off-diagonal element of $\mathbf{A}^T \mathbf{A}$. If the diagonal elements were not normal-

ized then this maximal value should be normalized with the diagonal values of this matrix. For the considered measurement matrix \mathbf{A} we get

$$\mu = 0.49$$

resulting in the sparsity limit

$$K < \frac{1}{2} \left(1 + \frac{1}{\mu} \right) \approx 1.5204.$$

The worst case, that determine value of μ , was coherence between 5th and 7th column of the considered matrix. The reconstruction is unique only for $K = 1$. Note that in contrast to the spark limit this condition will guarantee that the same unique solution is obtained using ℓ_1 -norm and ℓ_0 -norm. This is the reason why this limit is more conservative.

For a measurement matrix of order 5×8 the smallest possible value of the the coherence index is

$$\mu \geq \sqrt{\frac{N-M}{M(N-1)}} = \sqrt{\frac{3}{5 \times 7}} = 0.2928$$

with maximal possible bound $K < 2.2078$. The matrix with minimal coherence index is quite specific and it is called the equiangular tight frame (ETF). In practice many optimization approaches are based on finding the measurement matrix with coherence as low as possible (as close to the ETF as possible).

4.2.3 Restricted Isometry Property Constant Calculation

Restricted isometry property (RIP) constants of orders 1, 2, 3, 4, and 5 are calculated. For the calculation of the RIP constant with assumed sparsity $K = 1$ all possible submatrices \mathbf{A}_1 are formed. These are 5×1 matrices. There are 8 of them. The matrices $\mathbf{A}_1^T \mathbf{A}_1$ are formed. All of them are scalars equal to 1 with $\lambda = 1$, resulting in

$$\delta_1 = \max\{1 - \lambda_{\min}, \lambda_{\max} - 1\} = 0.$$

Next the sparsity $K = 2$ of the resulting \mathbf{X} is assumed. All possible measurement submatrices \mathbf{A}_2 corresponding to this sparsity are formed. There are $\binom{8}{2}$ of them. The matrices $\mathbf{A}_2^T \mathbf{A}_2$ are formed. Then their eigenvalues are calculated. The RIP constant δ_2 is obtained as a maximal value of

$$\delta_2 = \max\{1 - \lambda_{\min}, \lambda_{\max} - 1\} = 0.49.$$

over all possible submatrices \mathbf{A}_2 .

The calculation is repeated for assumed sparsity $K = 3, 4, 5$ by forming corresponding submatrices \mathbf{A}_3 , \mathbf{A}_4 , and \mathbf{A}_5 , respectively. The obtained numerical values for these sparsities are

$$\delta_3 = 0.9406$$

$$\delta_4 = 1.2063$$

$$\delta_5 = 1.3368$$

We can conclude that matrix \mathbf{A} satisfy the restricted isometry property

$$0 \leq \delta_K < 1$$

for sparsity 1, 2, and 3. The uniqueness condition require that for sparsity K measurement matrix satisfies restricted isometry property for $2K$ meaning that the uniqueness is guarantied only for $K = 1$. For $K = 2$ the condition should be satisfied for δ_4 , what is not the case.

The minimization for K sparse vector \mathbf{X} using ℓ_1 -norm will produce the same result as if ℓ_0 -norm were used if $\delta_{2K} < \sqrt{2} - 1$. It means that there is no guarantee that ℓ_1 norm minimization could be used in the reconstruction for sparsity $K = 1$. Note that different bounds have been derived in literature for this equivalence. One of the derived bounds is that $\delta_{2K} < 0.493$. The considered measurement matrix \mathbf{A} would produce a unique solution with ℓ_1 -norm based minimization, according to this bound since $\delta_2 = 0.49 < 0.493$.

From this example we can see that uniqueness conditions produce different limits, and that they are very restrictive.

4.3 Uniqueness of the DFT of Sparse Signals

In general, the reconstructed signal uniqueness is guarantied if the restricted isometry property is used and checked. However, two problems exist in the implementation of this approach. For a specific measurement matrix it produces quite conservative bounds. In addition, uniqueness check with the restricted isometry property requires a combinatorial approach, which is an NP hard problem.

In some reconstruction methods the missing measurements are considered as the minimization variables. The available measurements/samples are known and fixed. The number of variables in the minimization process is equal to the number of missing samples/measurements in the observation domain. This approach is possible when the common signal transforms are

the domains of signal sparsity [20, 48–54]. Then the missing and available samples/measurements form a complete set of samples/measurements.

The DFT is such a signal sparsity domain. The solution uniqueness is defined in the sense that the variation of the missing sample values cannot produce another signal of the same sparsity. In the case when the signal is already reconstructed then the uniqueness is checked in the sense that there is no other signal of the same or lower sparsity with the same set of available samples [20].

Consider a signal $x(n)$ with $n \in \mathbb{N} = \{0, 1, 2, \dots, N - 1\}$. Assume that Q of its samples at the positions $q_m \in \mathbb{N}_Q = \{q_1, q_2, \dots, q_Q\}$ are missing/omitted. The signal is sparse in the DFT domain, with sparsity K . The reconstruction goal is to get $x(n)$, for all $n \in \mathbb{N}$ using available samples at $n \in \mathbb{M} = \mathbb{N} \setminus \mathbb{N}_Q$. A new signal of the form

$$x_a(n) = x(n) + z(n)$$

will be analyzed here. For the available signal positions $n \in \mathbb{M}$ the value of $z(n)$ is fixed $z(n) = 0$, while $z(n)$ may take arbitrary value at the positions of missing samples $n = q_m \in \mathbb{N}_Q = \{q_1, q_2, \dots, q_Q\}$. If $x(n)$ is a K sparse signal then the DFT of $x_a(n)$ is

$$\begin{aligned} X_a(k) &= X(k) + Z(k) \\ &= N \sum_{i=1}^K A_i \delta(k - k_{0i}) + \sum_{m=1}^Q z(q_m) e^{-j2\pi q_m k / N}. \end{aligned}$$

Positions of nonzero values in $X(k)$ are $k_{0i} \in \mathbb{K} = \{k_{01}, k_{02}, \dots, k_{0K}\}$ with amplitudes $X(k_{0i}) = NA_i$. The values of missing samples of $x_a(n) = x(n) + z(n)$ for $n \in \mathbb{N}_Q$ are considered as variables. The goal of reconstruction process is to get $x_a(n) = x(n)$, or $z(n) = 0$ for all $n \in \mathbb{N}$. This goal should be achieved by minimizing a sparsity measure of the signal transform $X_a(k)$ [3, 47]. Existence of the unique solution of this problem depends on the number of missing samples, their positions, and the signal form.

If a signal with the transform $X(k)$ of sparsity K is obtained using a reconstruction method, with a set of missing samples, then the reconstruction $X(k)$ is unique if there is no other signal of the same or lower sparsity that satisfies the same set of available samples (using the same set of missing samples as variables).

Consider a signal $x(n)$ that is sparse in the DFT domain with unknown sparsity. Assume that the signal length is $N = 2^r$ samples and that Q samples are missing at the instants $q_m \in \mathbb{N}_Q$. Assume that the reconstruction is performed and that the

DFT of reconstructed signal is of sparsity K . The reconstruction result is unique if the inequality

$$K < N - \max_{h=0,1,\dots,r-1} \left\{ 2^h (Q_{2^h} - 1) \right\} - K$$

holds. Integers Q_{2^h} are calculated as

$$Q_{2^h} = \max_{b=0,1,\dots,2^h-1} \left\{ \text{card} \{ q : q \in \mathbb{N}_Q \text{ and } \text{mod}(q, 2^h) = b \} \right\} \quad (18)$$

For example, consider a signal with $N = 2^5 = 32$ and $Q = 9$ missing samples at

$$q_m \in \mathbb{N}_Q = \{2, 3, 8, 13, 19, 22, 23, 28, 30\}.$$

Using the presented we will find the sparsity limit K when we are able to claim that the reconstructed sparse signal is unique for any signal form.

- For $h = 0$ we use $Q_{2^0} = Q$ and get $2^0 (Q_{2^0} - 1) - 1 = (Q - 1) - 1 = 9$.
- For $h = 1$, the number Q_{2^1} is the greater value of

$$\text{card} \{ q : q \in \mathbb{N}_Q \text{ and } \text{mod}(q, 2) = 0 \} = \text{card} \{ 2, 8, 22, 28, 30 \} = 5$$

$$\text{card} \{ q : q \in \mathbb{N}_Q \text{ and } \text{mod}(q, 2) = 1 \} = \text{card} \{ 3, 13, 19, 23 \} = 4,$$

i.e., the maximal number of even or odd positions of missing samples. Thus $Q_{2^1} = \max \{ 5, 4 \} = 5$ with $2^1 (Q_{2^1} - 1) = 8$.

- Next Q_{2^2} is calculated as the maximal number of missing samples whose distance is multiple of 4. For various initial counting positions $b = 0, 1, 2, 3$ the numbers of missing samples with distance being multiple of 4 are 2, 1, 3, and 3, respectively. Then $Q_{2^2} = \max \{ 2, 1, 3, 3 \} = 3$ with $2^2 (Q_{2^2} - 1) = 8$.
- For Q_{2^3} the number of missing samples at distances being multiple of 8 are found for various $b = 0, 1, 2, 3, 4, 5, 6, 7$. The value of Q_{2^3} is 2 with $2^3 (Q_{2^3} - 1) = 8$.
- Finally we have two samples at distance 16 (samples at the positions $q_2 = 3$ and $q_5 = q_2 + N/2$) producing $Q_{2^4} = Q_{16} = 2$ with $2^4 (2 - 1) = 16$.

The reconstructed signal of sparsity K is unique if

$$K < N - \max_{h=0,1,2,3,4} \left\{ 2^h (Q_{2^h} - 1) \right\} - K$$

$$K < 32 - \max \{9, 8, 8, 8, 16\} - K$$

$$K < 32 - 16 - K$$

or

$$K < 8.$$

An extended discussion about the DFT uniqueness, within the framework of the missing samples as variables, can be found in [20].

5 Conclusion

Sparse signals can be reconstructed from a very reduced set of observations, through compressive sensing. This property has found applications in many fields. The topic of this paper was to introduce the basic definitions in compressive sensing. The conditions for exact and unique reconstruction of original signals are of crucial importance in theory and applications. These conditions are reviewed and related in this paper.

Appendix

MATLAB® functions for spark calculation (Algorithm 1), restricted isometry constant calculation (Algorithm 2) and uniqueness test for partial DFT matrix (Algorithm 4) are provided. Auxiliary function `nextcomb` used for generation of all possible columns combinations of the measurement matrix used in Algorithms 1 and 2 is given in Algorithm 3.

References

- [1] D. L. Donoho, "Compressed sensing," *IEEE Transactions on Information Theory*, vol. 52, no. 4, 2006, pp. 1289–1306.
- [2] E.J. Candès, M. Wakin, "An Introduction to Compressive Sampling", *IEEE Signal Processing Magazine*, vol. 25, no. 2, March 2008, pp. 21-30.
- [3] L. Stanković, M. Daković, S. Stanković, and I. Orović, "Sparse Signal Processing - Introduction," *Wiley Encyclopedia of Electrical and Electronics Engineering*, John Wiley & Sons, 2017.

Algorithm 1 Measurement matrix spark calculation

```

1  function s = spark(A)
2  % Matrix spark calculation
3  [M,N] = size(A);
4  s = M+1;
5  for k = 1:M
6      kk = nchoosek(N,k);
7      p = 1:k;
8      for m = 1:kk
9          A1 = A(:,p);
10         if rank(A1) < k
11             s = k; break
12         end
13         p = nextcomb(p,N);
14     end
15     if s < M+1, break, end
16 end

```

Algorithm 2 Restricted isometry constant calculation

```

1  function d = RIP_calc(A,K)
2  % Restricted isometry constant calculation
3  [M,N] = size(A);
4  d = 0;
5  kk = nchoosek(N,K);
6  p = 1:K;
7  for m = 1:kk
8      A1 = A(:,p);
9      l = eig(A1'*A1);
10     d = max([d, 1-min(l), max(l)-1]);
11     p = nextcomb(p,N);
12 end

```

Algorithm 3 Auxiliary function for generation of all combinations

```

1  function p = nextcomb(p,N)
2  % Generate next combination (in lexicographical order)
3  % Input: previous combination p and number of elements N
4  % Output: next combination or []
5  i = length(p);
6  K = N;
7  while i > 0 && p(i) == K
8      i = i-1;
9      K = K-1;
10 end
11 if i > 0
12     p(i) = p(i)+1;
13     for k = i+1:length(p)
14         p(k) = p(k-1)+1;
15     end
16 else
17     p = [];
18 end

```

Algorithm 4 Sparsity limit for partial DFT measurement matrix

```

1  function Kt = DFT_check(N,Nq)
2  % Sparsity limit for partial DFT matrix
3  % Inputs:
4  % N – total number of samples (must be power of two)
5  % Nq – set of missing sample positions
6  r = log2(N);
7  if r - round(r) ~= 0, error('N must be power of two'), end
8  Kt = N;
9  for h = 0:r-1
10     p = rem(Nq,2^h);
11     Q = zeros(1,2^h);
12     for s = 0:2^h-1
13         Q(s+1) = sum(p==s);
14     end
15     Q2h = max(Q);
16     if Kt > (N-2^h*(Q2h-1))/2
17         Kt = (N-2^h*(Q2h-1))/2;
18     end
19 end

```

- [4] E. J. Candès, J. Romberg, T. Tao, "Robust uncertainty principles: Exact signal reconstruction from highly incomplete frequency information," *IEEE Transactions on Information Theory*, vol. 52, no. 2, 2006, pp. 489-509.
- [5] Y. C. Eldar, G. Kutyniok, *Compressed Sensing: Theory and Applications*, Cambridge University Press, 2012.
- [6] R. Baraniuk, "Compressive sensing," *IEEE Signal Processing Magazine*, vol. 24, no. 4, 2007, pp. 118-121.
- [7] L. Stanković, S. Stanković, M. G. Amin, "Missing Samples Analysis in Signals for Applications to L-Estimation and Compressive Sensing", *Signal Processing*, Elsevier, vol. 94, Jan. 2014, pp. 401-408.
- [8] L. Stanković, *Digital Signal Processing with Applications: Adaptive Systems, Time-Frequency Analysis, Sparse Signal Processing*, CreateSpace Independent Publishing Platform, 2015.
- [9] S. Stanković, I. Orović, E. Sejdić, *Multimedia Signals and Systems: Basic and Advanced Algorithms for Signal Processing*, Springer International Publishing, second edition, 2015.
- [10] S. Stanković, L. Stanković, and I. Orović, "A Relationship between the Robust Statistics Theory and Sparse Compressive Sensed Signals Reconstruction," *IET Signal Processing*, 2014.
- [11] I. Orovic, and S. Stankovic, "Improved Higher Order Robust Distributions based on Compressive Sensing Reconstruction," *IET Signal Processing*, vol. 8, no. 7, 2014, pp. 738 - 748.
- [12] M. Elad, *Sparse and Redundant Representations: From Theory to Applications in Signal and Image Processing*, Springer, 2010.
- [13] I. Orović, V. Papić, C. Ioana, X. Li, and S. Stanković, "Compressive Sensing in Signal Processing: Algorithms and Transform Domain Formulations," *Mathematical Problems in Engineering*, 2016.
- [14] L. Stankovic, M. Dakovic, S. Vujovic, "Reconstruction of Sparse Signals in Impulsive Disturbance Environments," *Circuits, Systems and Signal Processing*, vol. 2016, pp. 1-28.
- [15] I. Stankovic, I. Orovic, M. Dakovic, and S. Stankovic, "Denoising of Sparse Images in Impulsive Disturbance Environment," *Multimedia Tools and Applications*, pp.1-21, 2017.
- [16] S. Stankovic, I. Orovic, and M. Amin, "L-statistics based Modification of Reconstruction Algorithms for Compressive Sensing in the Presence of Impulse Noise," *Signal Processing*, vol.93, no.11, November 2013, pp. 2927-2931.
- [17] E. J. Candès, "The restricted isometry property and its implications for compressed sensing", *Comptes Rendus Mathématique*, Vol.346, Issues 9-10, May 2008, pp 589-592.

-
- [18] T. Zhang, "Sparse Recovery with Orthogonal Matching Pursuit Under RIP," *IEEE Trans. on Information Theory*, 57(9), 2011, pp. 6215-6221.
- [19] D. L. Donoho, M. Elad, V. Temlyakov, "Stable recovery of sparse overcomplete representations in the presence of noise," *IEEE Transactions on Information Theory*, vol. 52, 2006, pp. 6-18.
- [20] L. Stanković, M. Daković, "On the Uniqueness of the Sparse Signals Reconstruction Based on the Missing Samples Variation Analysis," *Mathematical Problems in Engineering*, vol. 2015, Article ID 629759, 14 pages, doi:10.1155/2015/629759
- [21] L. Stankovic, M. Dakovic, T. Thayaparan, *Time-Frequency Signal Analysis with Applications*, Artech House, Boston, March 2013.
- [22] L. Stankovic, "On the ISAR Image Analysis and Recovery with Unavailable or Heavily Corrupted Data," *IEEE Transactions on Aerospace and Electronic Systems*, Vol.51, no.3, pp.2093-2106, July 2015
- [23] S. Costanzo, A. Rocha, M. D. Migliore, "Compressed Sensing: Applications in Radar and Communications", *The Scientific World Journal*, Volume 2016, Article ID 5407415, 2 pages, Editorial, 2016.
- [24] J. Ender, "On compressive sensing applied to radar", *Signal Processing*, Volume 90, Issue 5, May 2010, Pages 1402-1414.
- [25] L. Stankovic, S. Stankovic, T. Thayaparan, M. Dakovic, and I. Orovic, "Separation and Reconstruction of the Rigid Body and Micro-Doppler Signal in ISAR Part II - Statistical Analysis," *IET Radar, Sonar and Navigation*, vol.9, no.9, pp.1155-1161, 2015 doi: 10.1049/iet-rsn.2015.0137
- [26] L. Stankovic, S. Stankovic, T. Thayaparan, M. Dakovic, and I. Orovic, "Separation and Reconstruction of the Rigid Body and Micro-Doppler Signal in ISAR Part I- Theory," *IET Radar, Sonar and Navigation*, vol.9, no.9, pp.1147-1154, 2015 doi: 10.1049/iet-rsn.2015.0096
- [27] P. Flandrin, P. Borgnat, "Time-Frequency Energy Distributions Meet Compressed Sensing," *IEEE Transactions on Signal Processing*, vol.58, no.6, pp.2974, 2982, June 2010.
- [28] L. Stankovic, I. Orovic, S. Stankovic, and M. Amin, "Compressive Sensing Based Separation of Non-Stationary and Stationary Signals Overlapping in Time-Frequency," *IEEE Transactions on Signal Processing*, Vol. 61, no. 18, pp. 4562 - 4572, Sept. 2013.
- [29] I. Orović, S. Stanković, and T. Thayaparan, "Time-Frequency Based Instantaneous Frequency Estimation of Sparse Signals from an Incomplete Set of Samples," *IET Signal Processing, Special issue on Compressive Sensing and Robust Transforms*, 2014.
- [30] S. Stankovic, I. Orovic, M. Amin, "L-statistics based Modification of Reconstruction Algorithms for Compressive Sensing in the Presence of Impulse Noise," *Signal Processing*, vol.93, no.11, November 2013, pp. 2927-2931 , 2013.

- [31] G. Hua, Y. Hiang, G. Bi, "When Compressive Sensing meets Data Hiding", *IEEE Signal Processing Letters*, Vol. 23, No. 4, April 2016.
- [32] J. Bazerque, G. Giannakis, "Distributed spectrum sensing for cognitive radio networks by exploiting sparsity," *IEEE Trans. Signal Process.*, vol. 58, no. 3, pp. 1847-1862, Mar. 2010.
- [33] S. Stanković, and I. Orović, "An Approach to 2D Signals Recovering in Compressive Sensing Context," *Circuits, Systems and Signal Processing*, doi 10.1007/s00034-016-0366-8
- [34] I. Stanković, I. Orović, M. Daković, and S. Stanković, "Denoising of Sparse Images in Impulsive Disturbance Environment," *Multimedia Tools and Applications*, 2017, doi:10.1007/s11042-017-4502-7
- [35] M. Fornsaier, H. Rauhut, "Iterative thresholding algorithms", *Applied and Computational Harmonic Analysis*, vol. 25, no. 2, Sept. 2008, pp. 187- 208.
- [36] S. Mallat, Z. Zhang, "Matching pursuits with time-frequency dictionaries", *IEEE Transactions on Signal Processing*, vol. 41, no. 12, pp. 3397-3415, 1993.
- [37] J. A. Tropp, A. C. Gilbert, "Signal Recovery From Random Measurements Via Orthogonal Matching Pursuit," *IEEE Transactions on Information Theory*, Vol. 53, np. 12, pp. 4655-4666, 2007.
- [38] D. Needell, J. A. Tropp, "CoSaMP: Iterative signal recovery from noisy samples", *Applied and Computational Harmonic Analysis*, 2008. DOI: 10.1016/j.acha.2008.07.002.
- [39] R. Tibshirani, "Regression Shrinkage and Selection via the LASSO". *Journal of the Royal Statistical Society. Series B (methodological)* 58 (1). Wiley: 267-88, 1996.
- [40] S. Stanković, I. Orović, and L. Stanković, "An Automated Signal Reconstruction Method based on Analysis of Compressive Sensed Signals in Noisy Environment," *Signal Processing*, vol. 104, Nov 2014, pp. 43 - 50, 2014.
- [41] S. G. Mallat, Z. Zhang, "Matching pursuits with time-frequency dictionaries," *IEEE Transactions on Signal Processing*, vol. 41, no. 12, 1993, pp. 3397-3415.
- [42] I. Daubechies, M. Defrise, C. De Mol "An iterative thresholding algorithm for linear inverse problems with a sparsity constraint," *Communications on Pure and Applied Mathematics*, Volume 57, Issue 11, November 2004, Pages 1413-1457.
- [43] T. Blumensath, M. E. Davies, "Gradient Pursuits," *IEEE Transactions on Signal Processing*, vol.56, no.6, pp.2370-2382, June 2008,
- [44] M. A. Figueiredo, R. D. Nowak, S. J. Wright, "Gradient projection for sparse reconstruction: Application to compressed sensing and other inverse problems," *IEEE Journal of Selected Topics in Signal Processing*, vol. 1, no. 4, 2007, pp. 586-597.
- [45] E. Candès, J. Romberg, " ℓ_1 -magic : Recovery of Sparse Signals via Convex Programming, " Caltech, <http://users.ece.gatech.edu/justin/l1magic/downloads/l1magic.pdf>, Oct.2005.

-
- [46] L. Stanković, M. Daković, S. Vujović, "Adaptive Variable Step Algorithm for Missing Samples Recovery in Sparse Signals," *IET Signal Processing*, vol. 8, no. 3, 2014. pp. 246 -256.
- [47] L. Stanković, "A measure of some time–frequency distributions concentration," *Signal Processing*, vol. 81, 2001, pp. 621–631.
- [48] S. Stanković, I. Orović, and L. Stanković, "Polynomial Fourier Domain as a Domain of Signal Sparsity," *Signal Processing*, Vol. 130, Jan. 2017, pp. 243-253.
- [49] S. Stankovic, L. Stankovic, and I. Orovic, "Compressive sensing approach in the Hermite transform domain," *Mathematical Problems in Engineering*, Volume 2015 (2015), Article ID 286590, 9 pages <http://dx.doi.org/10.1155/2015/286590>.
- [50] A. Sandryhaila, J. Kovacevic, M. Puschel, "Compression of QRS complexes using Hermite expansion", *IEEE int. Conference on Acoust., Speech and Signal Process., ICASSP*, Prague, 2011, pp. 581-584.
- [51] M. Brajović, I. Orović, M. Daković, and S. Stanković, "On the Parameterization of Hermite Transform with Application to the Compression of QRS Complexes," *Signal Processing*, vol. 131, pp. 113-119, February 2017.
- [52] M. Brajović, I. Orović, M. Daković, and S. Stanković, "Gradient-based signal reconstruction algorithm in the Hermite transform domain," *Electronics Letters*, Vol. 52, no. 1, pp.41-43, 2016.
- [53] S. Stanković, I. Orović, and A. Krylov, "Video Frames Reconstruction based on Time-Frequency Analysis and Hermite projection method," *EURASIP Journal on Advances in Signal Processing, Special Issue on Time-Frequency Analysis and its Application to Multimedia signals*, Vol. 2010, Article ID 970105, 11 pages, 2010.
- [54] I. Orović, S. Stanković, T. Chau, C. M. Steele, and E. Sejdić, "Time-frequency analysis and Hermite projection method applied to swallowing accelerometry signals," *EURASIP Journal on Advances in Signal Processing*, Vol. 2010, Article ID 323125, 7 pages, 2010.