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DETERMINATION OF THE IMAGE FORCE ON A STRAIGHT DISLOCATION IN A HALF-SPACE BY THE BARNETT AND LOTHE FORMULA

A b s t r a c t

The Barnett and Lothe (1974) formula, expressed in terms of the prelogarithmic energy factors, is used to derive the classical expressions for the image force exerted on a straight dislocation by a free, slipping, or fixed boundary of an elastic and isotropic half-space. A dual analysis is presented in which the image force expressions are derived by the evaluation of the M integrals appearing in Rice's (1985) version of the Barnett and Lothe formula. The comparison with the classical derivations by Head (1953), Dundurs and Mura (1964), and Dundurs (1969), based on the complete solutions of the considered boundary-value problems, is discussed in each case.

Određivanje dislokacione sile u poluprostoru korišćenjem Barnett i Lothe-ove formule

I z v o d

Barnett i Lothe-ova (1974) formula, izražena korišćenjem predlogaritamskih energetskih faktora, je primijenjena u izvođenju klasičnih izraza za dislokacionu silu usljed dejstva slobodne, proklizavajuće ili fiksne konture elastičnog izotropnog poluprostora. Dualna analiza je data u kojoj su izrazi za dislokacionu silu izvedeni evaluacijom M integrala koji se pojavljuju u Rice-ovoj (1985) verziji Barnett i Lothe-ove formule. Upoređenje sa klasičnom

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analizom (Head, 1953; Dundurs and Mura, 1964; Dundurs, 1969), koja je zasnovana na rješavanju cijelog problema granične vrijednosti, je dato za sve tri vrste graničnih uslova.

1 Introduction

Barnett and Lothe (1974) formulated an image force theorem for dislocations in anisotropic bicrystals, according to which a straight dislocation with a Burgers vector $\{b_1, b_2, b_3\}$, where b_3 is its screw component, residing at a distance h from the interface (Fig. 1a), is under an energetic (image) force

$$f_1 = -(E^{\infty} - E^{1|2})/h\,. \tag{1.1}$$

In this formula, E^{∞} is the prelogarithmic energy factor for the same dislocation in an infinite homogeneous medium elastically identical to the half-space in which the dislocation resides, while $E^{1|2}$ is the prelogarithmic energy factor of the same dislocation located at the interface of the bicrystal under consideration (Fig. 1b). Barnett and Lothe (1974) also developed a procedure which allows the calculation of $E^{1|2}$ by numerical integration for any type of elastic anisotropy.

For an isotropic medium with shear modulus μ and Poisson's ratio ν, the prelogarithmic energy factor E^{∞} is well-known (e.g., Barnett and Lothe, 1974; Hirth and Lothe, 1982) and is given by

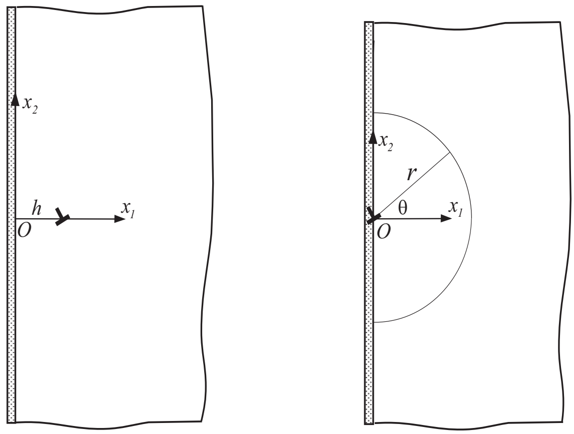
$$E^{\infty} = \frac{\mu(b_1^2 + b_2^2)}{4\pi(1-\nu)} + \frac{\mu b_3^2}{4\pi}\,, \tag{1.2}$$

because the elastic strain energy within the region of a large radius $R \gg \rho$ around the dislocation, outside the dislocation core of a small radius ρ, is then $U^{\infty} = E^{\infty} \ln(R/\rho)$.

From Rice's (1985) analysis it folows that the image force can also be calculated by an indirect evaluation of the J_1 integral, expressed in terms of the M integrals as

$$f_1 = J_1 = -(M^{\infty} - M^{1|2})/h\,. \tag{1.3}$$

In this formula, M^{∞} is the M integral around the dislocation in an infinite homogeneous medium elastically identical to the half-space in which the dislocation resides, defined with respect to the coordinate origin at the center of the dislocation, while $M^{1|2}$ is the M integral around the dislocation located at the interface of the bicrystal under consideration, defined with respect to the coordinate origin at the center of such interface dislocation. This follows because, if the circle around the dislocation is chosen to be large enough, the normal distance h between the dislocation and the interface is not observed, and the dislocation is seen from such large distances as an interface dislocation. The formula (1.3) neatly complements the Barnett and Lothe formula (1.1), expressed in terms of the prelogarithmic energy factors, and is equivalent to it because $M^{\infty} \equiv E^{\infty}$ and $M^{1|2} \equiv E^{1|2}$, although details of the calculation of M^{∞} and E^{∞}, or $M^{1|2}$ and $E^{1|2}$, are mathematically and conceptually different. For example, as shown in Appendix A, the M^{∞} integral is evaluated from



(a) (b)

Figure 1: (a) A straight dislocation at a distance h from a free, sliding, or fixed boundary ($x_1 = 0$) of an isotropic half-space. (b) A boundary dislocation, obtained from part (a) in the limit as $h \to 0$. The polar coordinates with the origin at the center of the dislocation are (r, θ).

$$M^{\infty} = \int_0^{2\pi} r^2 w^{\infty} \, \mathrm{d}\theta = \frac{\mu(b_1^2 + b_2^2)}{4\pi(1-\nu)} + \frac{\mu b_3^2}{4\pi}, \tag{1.4}$$

where w^{∞} is the strain energy density for the dislocation in an infinite homogeneous medium, and (r, θ) are the polar coordinates with the origin at the center of the dislocation.

We apply in this paper the Barnett and Lothe formula to determine the image force on a straight dislocation in a half-space with either traction-free ($\sigma_{11} = \sigma_{12} = \sigma_{13} = 0$), slipping ($u_1 = 0$, $\sigma_{12} = \sigma_{13} = 0$), or fixed ($u_1 = u_2 = u_3 = 0$) boundary (Fig. 1a). Toward this goal, and with the appropriately interpreted (1.1) and (1.3) in the case of a half-space rather than a bicrystal configuration, it is sufficient to determine $E^{1|2}$ or $M^{1|2}$ for each boundary condition. In the evaluation of $M^{1|2}$, the integral is evaluated along a semicircle around the boundary dislocation, because the contribution to M integral from the edge $x_1 = 0$ identically vanishes for all three considered boundary conditions. It will be shown that for a slipping and fixed boundary, the $M^{1|2}$ integral can be determined from (Fig. 1b)

$$M^{1|2} = \int_{-\pi/2}^{\pi/2} r^2 w^{1|2} \, \mathrm{d}\theta, \tag{1.5}$$

with $w^{1|2}$ designating the corresponding strain energy density. For a traction-free boundary, $M^{1|2} = 0$, because $w^{1|2} = 0$, trivially. For a slipping boundary of a half-space, the complete elasticity solution can be derived by a simple superposition of the infinite-medium elastic fields of the dislocation under consideration and its properly defined image dislocation. The elastic stress field needed for the evaluation of $w^{1|2}$ in the case of a fixed boundary of a half-space is derived from the corresponding Airy stress function, as shown in Appendix B. Appendix A contains a necessary background for the derivation of Rice's version of the Barnett and Lothe formula.

The presented analysis will be related to the liming cases of the classical analyzes by Head (1953), Dundurs and Mura (1964), and Dundurs (1969), which are based on the complete solutions of the considered boundary-value problems, and the use of the Peach–Koehler (1950) dislocation force expression, or the negative gradient of the strain energy with respect to the dislocation position. This comparative analysis is appealing from both the conceptual and the methodological point of view.

2 Traction-free boundary

When the dislocation is located at a traction-free boundary of a half-space, it has exited the material and no strain energy remains in a half-space. Thus, $U^{1|2}$ vanishes and so do $E^{1|2}$ and $M^{1|2}$. Consequently from (1.1) and (1.3), the image force exerted on a dislocation at a distance h from a free boundary of a half-space is

$$f_1 = -\frac{E^\infty}{h} = -\frac{M^\infty}{h} = -\frac{\mu(b_1^2+b_2^2)}{4\pi h(1-\nu)} - \frac{\mu b_3^2}{4\pi h}\,, \tag{2.1}$$

regardless of the orientation of the edge component of the Burgers vector. It can also be easily shown that the radial component of the dislocation force toward the tip of a wedge of any angle is also equal to E^∞/h (Asaro, 1975; Rice, 1985).

The expression on the right-hand side of (2.1) was originally derived for an edge dislocation be Head (1953). It can also be deduced from the general results by Dundurs and Mura (1964), and Dundurs (1969) for a straight dislocation near perfect interface between two half-spaces with different elastic constants, by taking the shear modulus of a half-space in which the dislocation does not reside to be equal to zero. The expression for the strain energy of a straight dislocation (including its screw component) near a bimaterial interface is given by eq. (8) of Lubarda (1997). When $\mu_2 \to 0$, this expression reduces to

$$U = \frac{\mu}{4\pi(1-\nu)}\left[b_1^2+b_2^2+(1-\nu)b_3^2\right]\ln\frac{2h}{\rho} - \frac{\mu}{16\pi(1-\nu)^2}\left[(3-4\nu)b_1^2 - b_2^2\right]. \tag{2.2}$$

By taking the negative gradient of U with respect to h, we recover the right-hand side of (2.1). Furthermore, it follows that $-\rho\partial U/\partial\rho = M^\infty$, as given in (1.4).

2.1 Direct evaluation of J_1

It is appealing to compare the indirect evaluation of the J_1 integral, via the M integrals appearing in (1.3), with the direct evaluation of the J_1 integral. By taking a closed contour to consist of the segment $(-R, R)$ along a free boundary and a semicircle of radius R, centered at O (5a), it follows that, in the limit as $R \to \infty$, the non-vanishing contribution to J_1 integral comes only from the stresses along the boundary $x_1 = 0$, such that

$$J_1 = -\frac{1}{2\mu}\left[(1-\nu)\int_0^\infty \sigma_{22}^2(0,x_2)\,\mathrm{d}x_2 + 2\int_0^\infty \sigma_{32}^2(0,x_2)\,\mathrm{d}x_2\right]. \tag{2.3}$$

The stress components $\sigma_{22}(0,x_2)$ and $\sigma_{32}(0,x_2)$ along a free surface $x_1 = 0$ can be determined solely in terms of the infinite-medium stresses, without solving the entire boundary-value problem, by employing some general results from the two-dimensional elasticity (Kienzler and Duan, 1987; Lin, Honein and Herrmann, 1989; Lubarda, 2015,2016). This gives

$$\sigma_{22}(0,x_2) = 2[\sigma_{22}^\infty(0,x_2) - \sigma_{11}^\infty(0,x_2)]\,, \quad \sigma_{32}(0,x_2) = 2\sigma_{32}^\infty(0,x_2)\,. \tag{2.4}$$

Using the well known expressions for the infinite-medium dislocation stresses σ_{ij}^∞ (e.g., Hirth and Lothe, 1982), it follows that

$$\sigma_{22}(0,x_2) = \frac{4\mu h x_2}{\pi(1-\nu)}\,\frac{b_1 h - b_2 x_2}{(h^2+x_2^2)^2}\,, \quad \sigma_{32}(0,x_2) = -\frac{\mu b_3}{\pi}\,\frac{h}{h^2+x_2^2}\,. \tag{2.5}$$

The substitution of (2.5) into (2.3) and integration reproduces (2.1).

3 Slipping boundary

Figure 3a shows a straight dislocation with a Burgers vector $\{b_1, b_2, b_3\}$ at a distance h from a slipping boundary of a half-space, which prevents normal displacement ($u_1 = 0$, $\sigma_{11} \neq 0$), but allows tangential displacements ($u_2 \neq 0$, $u_3 \neq 0$, $\sigma_{12} = \sigma_{13} = 0$). The dislocation is created by the displacement discontinuity along the x_1 axis from $x_1 = h$ to infinity. The complete elastic solution for this problem can be obtained by placing in an infinite medium an image dislocation with a Burgers vector $\{b_1, -b_2, -b_3\}$ at point ($x_1 = -h$, $x_2 = 0$, Fig. 3a), because the two dislocations together produce neither shear stress nor horizontal displacement along the plane $x_1 = 0$ in an infinite medium ($\sigma_{12} = \sigma_{13} = 0\,, u_1 = 0$).

If $h \to 0$ in Fig. 3a, the dislocation resides at the boundary $x_1 = 0$, Fig. 2b. The image dislocation in an infinite medium then merges with the actual dislocation to produce a dislocation with a Burgers vector $\{2b_1, 0, 0\}$. Physically, a slipping boundary cannot support the screw component of the dislocation, nor the edge component parallel to the boundary. The strain energy in a half-space shown in Fig. 3b, stored in the region between a semicircular core of radius ρ and a large semicircle of radius R, is thus one-half of the strain energy in an infinite medium due to dislocation of a Burgers vector $2b_1$, stored in the region between the circles of radii ρ and R, i.e.,

$$U^{1|2} = \frac{1}{2}U_{2b_1}^\infty\,, \quad U_{2b_1}^\infty = \frac{\mu}{4\pi(1-\nu)}\,(2b_1)^2\,\ln\frac{R}{\rho}\,. \tag{3.1}$$

The work done by the tractions on the corresponding displacements over the core semicircle of radius ρ and a remote semicircle of radius R cancel each other ($W_\rho = -W_R$), which was used in arriving at (3.1). Thus,

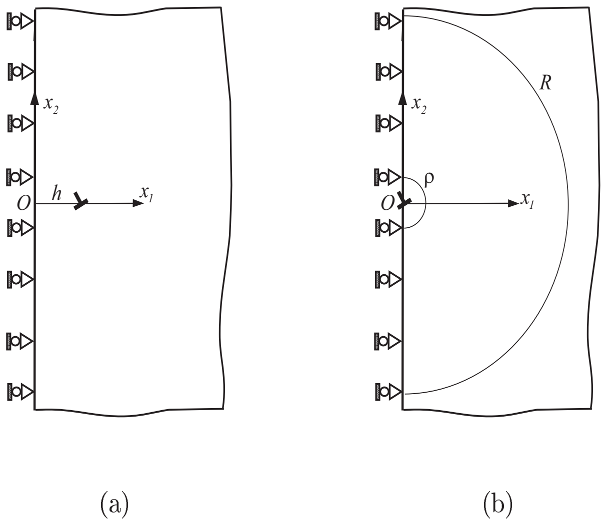


Figure 2: (a) A straight dislocation with a Burgers vector $\{b_1, b_2, b_3\}$ at a distance h from the sliding boundary of an isotropic half-space. (b) A boundary dislocation, obtained from part (a) in the limit as $h \to 0$. The elastic strain energy is calculated in the region between a semicircular core of small radius ρ and a large semicircle of radius R.

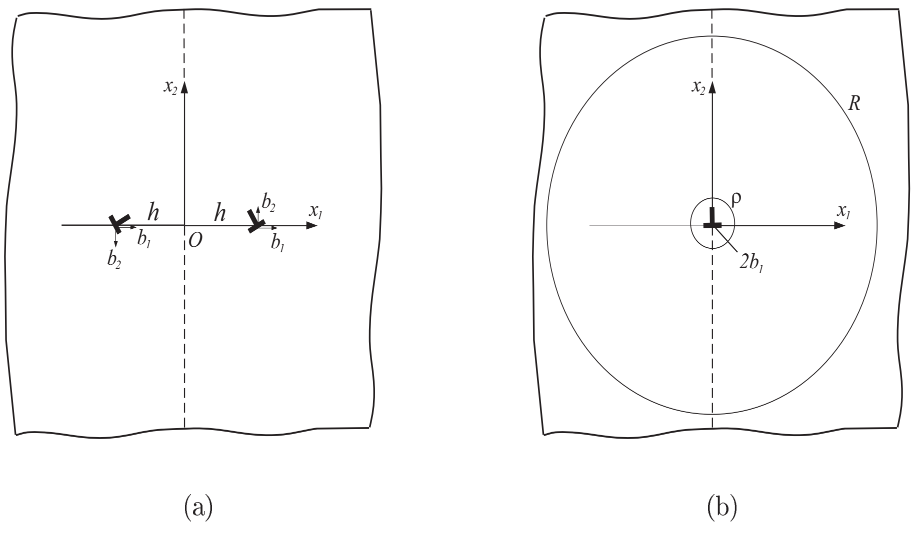


Figure 3: (a) The solution to the problem in Fig. 2a is obtained by superimposing the infinite-medium elastic fields of the actual dislocation with a Burgers vector $\{b_1, b_2, b_3\}$ and an image dislocation with a Burgers vector $\{b_1, -b_2, -b_3\}$, placed at point $(x_1 = -h,\, x_2 = 0.$ (b) In the limit as $h \to 0$, the image dislocation in Fig. 3a merges with the actual dislocation to produce a dislocation with a Burgers vector $\{2b_1, 0, 0\}$.

$$U^{1|2} = E^{1|2} \ln \frac{R}{\rho} , \quad E^{1|2} = \frac{\mu b_1^2}{2\pi(1-\nu)} . \tag{3.2}$$

The substitution of E^{∞} from (1.2) and $E^{1|2}$ from (3.2) into the Barnett and Lothe formula (1.1) therefore gives

$$f_1 = \frac{\mu(b_1^2 - b_2^2)}{4\pi h(1-\nu)} - \frac{\mu b_3^2}{4\pi h} . \tag{3.3}$$

The dislocation is in equilibrium for any $h > \rho$ if $b_1^2 = b_2^2 + (1-\nu)b_3^2$. For example, a pure edge dislocation is in equilibrium if its Burgers vector is oriented so that $b_1 = \pm b_2$.

3.1 M integral derivation

We only need to evaluate $M^{1|2}$ and use (1.3). The $M^{1|2}$ integral is simply equal to one-half of the M^{∞} for the dislocation of the Burgers vector $2b_1$ in an infinite medium. Thus, by using (1.4),

$$M^{1|2} = \int_{-\pi/2}^{\pi/2} r^2 w^{1|2} \, \mathrm{d}\theta = \frac{1}{2} M^{\infty}_{2b_1} = \frac{1}{2} \frac{\mu(2b_1)^2}{4\pi(1-\nu)} = \frac{\mu b_1^2}{2\pi(1-\nu)} . \tag{3.4}$$

The substitution of M^{∞} from (1.4) and $M^{1|2}$ from (3.4) into (1.3) reproduces the dislocation force expression (3.3).

3.2 Comparison with other derivations

The expression (3.3) can be easily verified by using the Peach–Koehler (1950) expression

$$f_1 = \hat{\sigma}_{21} b_1 + \hat{\sigma}_{22} b_2 + \hat{\sigma}_{23} b_3 . \tag{3.5}$$

The stress components $\hat{\sigma}_{ij}$ at the dislocation position $x_1 = h$ are equal to those produced in an infinite medium by the image dislocation $\{b_1, -b_2, -b_3\}$ at $x_1 = -h$. These are

$$\hat{\sigma}_{21} = \frac{\mu b_1}{2\pi(1-\nu)} \frac{1}{2h} , \quad \hat{\sigma}_{22} = -\frac{\mu b_2}{2\pi(1-\nu)} \frac{1}{2h} , \quad \hat{\sigma}_{23} = -\frac{\mu b_3}{2\pi} \frac{1}{2h} . \tag{3.6}$$

The substitution of (3.6) into (3.5) reproduces (3.3).

Alternatively, (3.3) can be derived as the negative gradient of the strain energy with respect to a dislocation position, which is the approach frequently used by Dundurs (1969), among others. Thus,

$$f_1 = -\frac{\partial U}{\partial h} , \quad U = \frac{1}{2} \int_{h+\rho}^{R} (\sigma_{21} b_1 + \sigma_{22} b_2 + \sigma_{23} b_3)_{x_2=0} \, \mathrm{d}x_1 + W_\rho + W_R . \tag{3.7}$$

The terms W_ρ and W_R (independent of h) account for the work done by tractions on the corresponding displacements over the core circle and a remote semicircle. The stress components along the x_1 axis in Fig. 3a are the sums of the contributions from the actual and image dislocation in an infinite medium, i.e.,

$$(\sigma_{21}, \sigma_{22}) = \frac{\mu b_1}{2\pi(1-\nu)} \left(\frac{1}{x_1 - h} \pm \frac{1}{x_1 + h} \right), \quad \sigma_{23} = \frac{\mu b_3}{2\pi} \left(\frac{1}{x_1 - h} - \frac{1}{x_1 + h} \right), \tag{3.8}$$

so that

$$U = \frac{\mu}{4\pi(1-\nu)} \left(b_1^2 \ln \frac{R^2}{2h\rho} + b_2^2 \ln \frac{2h}{\rho} \right) + \frac{\mu b_3^2}{4\pi} \ln \frac{2h}{\rho} + W_\rho + W_R \,. \tag{3.9}$$

Using (3.9) in (3.7) reproduces (3.3). Also, (3.3) can be deduced from the analysis of edge dislocation near an inclusion with a slipping interface by Dundurs and Gangadharan (1969), by letting the radius of a rigid inclusion to be much greater than the distance of the dislocation from the interface; their expression (17) and (18) then specify the edge dislocation contribution to the image force. The screw dislocation contribution is the same as in the case of a traction-free boundary. For an analysis of dislocation interaction with interfaces whose properties are in between slipping and fixed interface properties, see the analyses by Shilkrot and Srolovitz (1998), and Fan and Wang (2003).

The total strain energy in a half-space with a slipping boundary (Fig. 3a), stored within a semicircle of large radius R with the center at O, outside a circular core of small radius ρ, is given by (3.9), in which

$$W_\rho = -\frac{1}{2} \int_0^{2\pi} (\sigma_r u_r + \sigma_{r\theta} u_\theta)_{r=\rho}\, \rho \, \mathrm{d}\theta \,, \quad W_R = \frac{1}{2} \int_{-\pi/2}^{\pi/2} (\sigma_r u_r + \sigma_{r\theta} u_\theta)_{r=R}\, R \, \mathrm{d}\theta \tag{3.10}$$

account for the work done by tractions on the corresponding displacements over the core circle and a remote semicircle. No work is done by the boundary traction along $x_1 = 0$. For a sufficiently small core radius, the work W_ρ is equal to that for a dislocation in an infinite medium, while W_R is equal to one-half of the work along the entire circle of large radius R around the dislocation of a Burgers vector $2b_1$ in an infinite medium. Thus, by using the expressions from Asaro and Lubarda (2006), page 425,

$$W_\rho = \frac{\mu}{8\pi(1-\nu)^2} \left[(1-\nu)(b_1^2 - b_2^2) - \frac{1}{2}\,(b_1^2 + b_2^2) \right], \quad W_R = \frac{\mu(1-2\nu)}{8\pi(1-\nu)^2}\, b_1^2 \,. \tag{3.11}$$

It is noted that in (3.9), $-\rho \partial U/\partial \rho = M^\infty$, which is the relationship that holds for all three boundary conditions of a half-space, consistent with the physical interpretation of M^∞ integral as the energy release rate associated with a self-similar expansion of a dislocation core.

4 Fixed boundary

Figure 4a shows a straight dislocation with a Burgers vector $\{b_1, b_2, b_3\}$ at a distance h from a fixed boundary of a half-space, which prevents all three displacement components

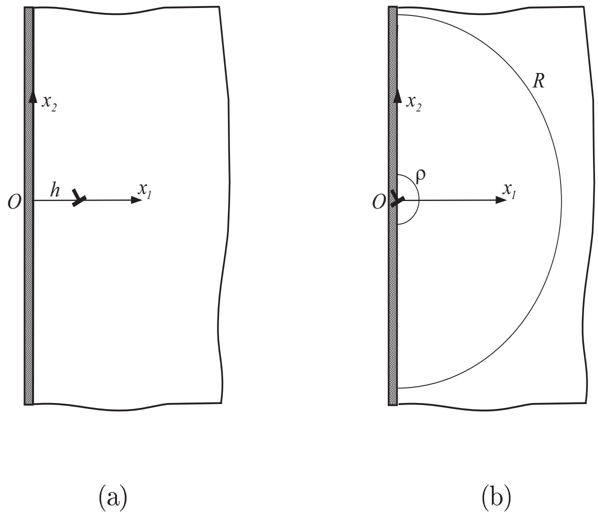


Figure 4: (a) A straight dislocation at a distance h from a fixed boundary of an isotropic half-space. (b) A boundary dislocation, obtained from part (a) in the limit as $h \to 0$. The elastic strain energy is calculated in the region between a semicircular core of small radius ρ and a large semicircle of radius R.

$(u_1 = u_2 = u_3 = 0)$. To determine the image force, it is sufficient to consider the dislocation residing at the boundary $(h = 0)$. The stress and displacement fields for such dislocation are listed in Appendix B of the paper. From this, the elastic strain energy within a large semicircle of radius R, outside a semicircular core of small radius ρ (Fig. 4b), is

$$U^{1|2} = \frac{1}{2}\int_\rho^R (\sigma_{r\theta} b_1 + \sigma_\theta b_2 + \sigma_{3\theta} b_3)^{1|2}_{\theta=0}\, \mathrm{d}r\,. \tag{4.1}$$

Since

$$(\sigma_{r\theta})^{1|2}_{\theta=0} = \frac{4\mu(1-\nu)}{\pi(3-4\nu)}\,\frac{b_1}{r}\,, \quad (\sigma_\theta)^{1|2}_{\theta=0} = \frac{4\mu(1-\nu)}{\pi(3-4\nu)}\,\frac{b_2}{r}\,, \quad \sigma^{1|2}_{3\theta} = \frac{\mu b_3}{\pi r}\,, \tag{4.2}$$

as shown in Appendix B, there follows

$$U^{1|2} = E^{1|2} \ln\frac{R}{\rho}\,, \quad E^{1|2} = \frac{2\mu(1-\nu)(b_1^2+b_2^2)}{\pi(3-4\nu)} + \frac{\mu b_3^2}{2\pi}\,, \tag{4.3}$$

where $E^{1|2}$ is the corresponding prelogarithmic factor. Expression for $E^{1|2}$ in (4.3) coincides with the Barnett and Lothe (1995) expression (14) if in their expression $\mu_2 \to \infty$.

The substitution of E^∞ from (1.2) and $E^{1|2}$ from (4.3) into the Barnett and Lothe formula (1.1) gives the following expression for the image force on a dislocation at a distance h from a fixed boundary (Fig. 4a),

$$f_1 = \frac{\mu(b_1^2+b_2^2)}{4\pi h(3-4\nu)}\left[4(1-2\nu) + \frac{1}{1-\nu}\right] + \frac{\mu b_3^2}{4\pi h}\,. \tag{4.4}$$

As in the case of a traction-free boundary, this force is independent of the direction of the edge component of the dislocation. Ting and Barnett (1993) also derived an expression for the image force on a dislocation in an arbitrary anisotropic half-space with a fixed boundary.

4.1 M integral derivation

From the results in Appendix B, the strain energy density is

$$w^{1|2} = \left[\frac{4(1-\nu)\mu(b_1^2+b_2^2)}{\pi^2(3-4\nu)^2}\left(1-2\nu+\cos^2\theta\right)+\frac{\mu b_3^2}{2\pi^2}\right]\frac{1}{r^2}, \tag{4.5}$$

so that

$$M^{1|2} = \int_{-\pi/2}^{\pi/2} r^2 w^{1|2}\,\mathrm{d}\theta = \frac{2(1-\nu)\mu(b_1^2+b_2^2)}{\pi(3-4\nu)}+\frac{\mu b_3^2}{2\pi}\,. \tag{4.6}$$

The substitution of M^∞ from (1.4) and $M^{1|2}$ from (4.6) into (1.3) reproduces (4.4).

4.2 Comparison with other derivations

The derived expression (4.4) can also be deduced from the general results of Head (1953), Dundurs and Mura (1964), and Dundurs (1969) for a straight dislocation near a fixed interface between two half-spaces with different elastic constants, by taking the shear modulus of a half-space in which the dislocation does not reside to be infinitely large. Their expressions were deduced from the Peach-Koehler force expression, or from the negative gradient of the interaction energy with respect to the dislocation position, once the entire boundary value problem was solved. A complete expression for the strain energy of the dislocation (including its screw component) near a bimaterial interface is given by eq. (8) of Lubarda (1997). When $\mu_2 \to \infty$, this expression reduces to

$$U = \frac{\mu(b_1^2+b_2^2)}{4\pi(3-4\nu)}\left[8(1-\nu)\ln\frac{R}{\rho}-\frac{5-12\nu+8\nu^2}{1-\nu}\ln\frac{2h}{\rho}\right]+\frac{\mu b_3^2}{4\pi}\left(\ln\frac{R}{\rho}-\ln\frac{2h}{\rho}\right)+W\,, \tag{4.7}$$

with the appropriately specified work contribution $W = W_R + W_\rho$, which is independent of h. By taking the negative gradient of U with respect to h, we recover $f_1 = -\partial U/\partial h$ as specified in (4.4). Furthermore, it follows that $-\rho\partial U/\partial\rho = M^\infty$, as given in (1.4), while the prelogarithmic factor in front of $\ln(R/\rho)$ term in (4.7) is $E^{1|2}$, as given in (4.3).

5 Conclusions

We have derived in this paper the classical expressions for the image force exerted on a straight dislocation by a traction-free, slipping, or fixed boundary of an isotropic half-space (Head, 1953; Dundurs and Mura, 1964; Dundurs, 1969) by using either the Barnett and Lothe (1974) formula $f_1 = -(E^\infty - E^{1|2})/h$, expressed in terms of the prelogarithmic energy factors E^∞ and $E^{1|2}$, or an equivalent formula $f_1 = -(M^\infty - M^{1|2})/h$, expressed in terms of the M integrals, which is deduced from Rice's (1985) analysis of a dislocation near bimaterial interface. The normal distance between a dislocation and a boundary of a half-space is h, E^∞ and M^∞ are the quantities evaluated for the dislocation in an infinite medium, while $E^{1|2}$ and $M^{1|2}$ correspond to a dislocation located at a boundary of a half-space. Although the two formulas are equivalent in the sense that $M^\infty \equiv E^\infty$ and $M^{1|2} \equiv E^{1|2}$, the details of the evaluations of M^∞ and E^∞, or $M^{1|2}$ and $E^{1|2}$, are mathematically and conceptually different, as explicitly demonstrated in this paper. Since the expressions for E^∞ and M^∞ are well known, only the expressions for $E^{1|2}$ and $M^{1|2}$ needed to be evaluated for each of the three considered boundary conditions. It is shown that the $M^{1|2}$ integral is the angle integral of $r^2 w^{1|2}$ along a semicircle of radius r around the boundary dislocation, where $w^{1|2}$ is the corresponding strain energy density. For a traction-free boundary, $M^{1|2} = 0$, because $w^{1|2} = 0$. For a slipping boundary, $M^{1|2}$ integral for a dislocation with a Burgers vector $\{b_1, b_2, b_3\}$ is equal to one-half of M^∞ for a dislocation with a Burgers vector $\{2b_1, 0, 0\}$ because $w^{1|2} = w^\infty_{2b_1}$. This is so because the complete elasticity solution for a dislocation with a Burgers vector $\{b_1, b_2, b_3\}$ near a slipping boundary of a half-space is the sum of the infinite-medium solutions for the actual dislocation and an image dislocation with a Burgers vector $\{b_1, -b_2, -b_3\}$. For a fixed boundary, $w^{1|2}$ is evaluated from the elastic field of a boundary dislocation, which is conveniently deduced from the corresponding Airy stress function.

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Appendix A: M integral

The M integral is defined by (Knowles and Sternberg, 1972; Budiansky and Rice, 1973)

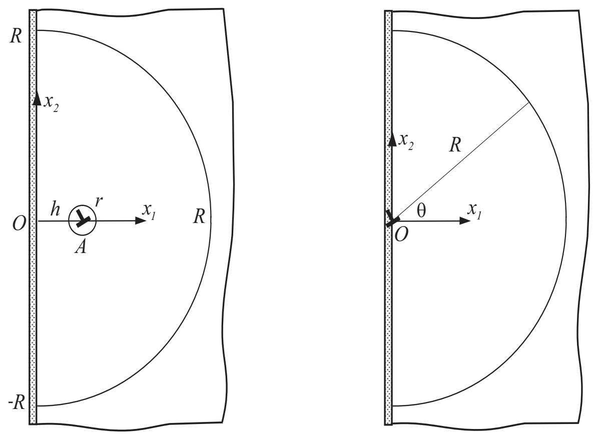
$$M = \oint P_{\alpha\beta} n_\alpha x_\beta \, \mathrm{d}l \,, \quad (\alpha, \beta) = 1, 2 \,, \tag{A.1}$$

where the integration is over a closed contour whose infinitesimal element is $\mathrm{d}l$, with the outward normal n_α. The components of the energy momentum tensor $P_{\alpha\beta}$ (Eshelby, 1956) are

$$P_{\alpha\beta} = w\delta_{\alpha\beta} - \sigma_{\alpha\gamma} u_{\gamma,\beta} - \sigma_{\alpha 3} u_{3,\beta} \,, \quad w = \frac{1}{2} \sigma_{\alpha\beta} \epsilon_{\alpha\beta} + \sigma_{3\gamma} \epsilon_{3\gamma} \,. \tag{A.2}$$

The strain energy density is denoted by w, $\delta_{\alpha\beta}$ are the components of the Kronecker delta tensor, and the comma specifies the indicated partial derivative. The summation convention is implied over a repeated index. The strain components are related to stress components by Hooke's law $\epsilon_{\alpha\beta} = (\sigma_{\alpha\beta} - \nu\sigma_{\gamma\gamma}\delta_{\alpha\beta})/2\mu$ and $\epsilon_{3\gamma} = \sigma_{3\gamma}/2\mu$, where μ is the elastic shear modulus, ν is the Poisson ratio, and $\sigma_{\gamma\gamma} = \sigma_{11} + \sigma_{22}$.

The value of M integral depends on the selected coordinate origin. The M integral with respect to the coordinate origin at point O is related to the M integral with respect to the



(a) (b)

Figure 5: (a) A straight dislocation at point A, a distance h from the boundary of an isotropic half-space. The M_A integral can be conveniently evaluated along a small circle of radius $r \ll h$ around A. The M_O integral around a dislocation can be conveniently evaluated along a closed contour which consists of a long boundary segment from $x_1 = -R$ to $x_1 = R$, completed by a large semicircle of radius $R \gg h$. The contribution to M_O from the $(-R, R)$ boundary segment vanishes, while along a large semicircle of radius R the distance h is not observed and a dislocation acts as a boundary dislocation shown in part (b).

coordinate origin at point A, at distance h from point O along x_1 axis, by

$$M_A = M_O - hJ_1 \,, \quad J_1 = \oint P_{\alpha 1} n_\alpha \, \mathrm{d}l \,. \tag{A.3}$$

Here, J_1 integral, evaluated around a defect (dislocation), represents the energy release rate associated with an infinitesimal translation of the defect in the x_1 direction. Thus, J_1 integral can be used to evaluate the dislocation force from $f_1 = J_1 = -(M_A - M_O)/h$.

For the problem of a dislocation near a free, slipping, or fixed boundary of a half-space (Fig. 5a), the integral M_A can be conveniently evaluated along a small circle of radius $r \ll h$ around A. The stress and displacement fields along such circle are nearly those of a dislocation in an infinite medium. If they are substituted into the expression for M_A integral, which can be expressed in polar coordinates with the origin at A, as

$$M_A = \oint [w - (\sigma_r u_{r,r} + \sigma_{r\theta} u_{\theta,r} + \sigma_{r3} u_{3,r})]^{\infty} \, r^2 \, \mathrm{d}\theta \,, \tag{A.4}$$

there follows (e.g., Rice, 1985)

$$M_A = \frac{\mu(b_1^2 + b_2^2)}{4\pi(1-\nu)} + \frac{\mu b_3^2}{4\pi} \,. \tag{A.5}$$

Indeed, the strain energy density for a dislocation in an infinite medium is

$$w^{\infty} = \left\{ \frac{1}{2\pi}\, E^{\infty} + \frac{\mu\nu}{8\pi^2(1-\nu)}\, \left[(b_1^2 - b_2^2)\cos 2\theta + 2b_1 b_2 \sin 2\theta \right] \right\} \frac{1}{r^2}. \tag{A.6}$$

Since

$$r^2(\sigma_r u_{r,r} + \sigma_{r\theta} u_{\theta,r} + \sigma_{r3} u_{3,r})^{\infty} = -\frac{\mu(b_1^2 + b_2^2)(1-2\nu)}{8\pi^2(1-\nu)^2}\, \cos 2\theta\,, \tag{A.7}$$

whose integral from 0 to 2π is equal to zero, the M_A integral is equal to the contour integral of $r^2 w^{\infty}$, which gives (A.5).

To evaluate M_O integral around a dislocation in Fig. 5b, we conveniently choose for the contour a long boundary segment from $x_1 = -R$ to $x_1 = R$, completed by a large semicircle of radius $R \gg h$. The contribution to M_O from $(-R, R)$ boundary segment vanishes for each of the three considered boundary conditions, because $n_2 = 0$ and $P_{12} = -\sigma_{1\gamma} u_{\gamma,2} - \sigma_{13} u_{3,2} = 0$ along the edge $x_1 = 0$. Along a large semicircle of radius $R \gg h$ centered at O, the distance h is not observed and the dislocation acts as if was an interface (boundary) dislocation. Thus, we evaluate M_O from

$$M_O = \int_{-\pi/2}^{\pi/2} [w - (\sigma_r u_{r,r} + \sigma_{r\theta} u_{\theta,r} + \sigma_{r3} u_{3,r})]^{1|2}\, R^2\, \mathrm{d}\theta\,. \tag{A.8}$$

The attached superscript $(\,)^{1|2}$ indicates that the integrand in (A.8) is evaluated for the elasticity field of a boundary dislocation from Fig. 5b. It is shown bellow that for a slipping or fixed boundary of a half-space, only w-part of the integrand in (A.8) contributes to M_O integral, so that (A.8) reduces to

$$M_O = \int_{-\pi/2}^{\pi/2} R^2 w^{1|2}\, \mathrm{d}\theta\,. \tag{A.9}$$

For a free boundary of a half-space, $w^{1|2} = 0$ and thus $M^{1|2} = 0$.

A1. Dislocation at a slipping boundary

For a dislocation at a slipping boundary (Fig. 3b), the elastic field in a half-space is equivalent to an infinite medium field to the right of a dislocation with a Burgers vector $2b_1$ (Fig. 3b). Thus,

$$w^{1|2} = w_{2b_1}^{\infty} = \left[\frac{1}{2\pi}\, E_{2b_1}^{\infty} + \frac{\mu\nu}{8\pi^2(1-\nu)}\, b_1^2 \cos 2\theta \right] \frac{1}{r^2}\,. \tag{A.10}$$

Furthermore, since

$$r^2(\sigma_r u_{r,r} + \sigma_{r\theta} u_{\theta,r} + \sigma_{r3} u_{3,r})^{1|2} = -\frac{\mu(2b_1)^2(1-2\nu)}{8\pi^2(1-\nu)^2}\, \cos 2\theta\,, \tag{A.11}$$

whose integral from $-\pi/2$ to $\pi/2$ is equal to zero, only the w-part of the integrand in (A.8) contributes to M_O integral, giving rise to (A.9).

A2. Dislocation at a fixed boundary

In this case, $(\sigma_r u_{r,r} + \sigma_{r\theta} u_{\theta,r} + \sigma_{r3} u_{3,r})^{1|2} = 0$, because it is shown in Appendix B that for a dislocation at a fixed boundary (Fig. 4b), the displacement components u_r, u_θ, and u_3 are all independent of r. Thus, again, only the w-part of the integrand in (A.8) contributes to M_O integral, which, therefore, reduces to (A.9).

Appendix B: Elastic fields for a straight dislocation at a fixed boundary of a half-space

The complete stress, strain and displacement fields are listed in this appendix for a straight dislocation with a Burgers vector $\{b_1, b_2, b_3\}$ residing at a fixed boundary of an isotropic half-space with the shear modulus μ and Poisson's ratio ν (Fig. 4b). For convenience, the results are given separately for each component of the dislocation Burgers vector. These fields can be deduced from the results by Dundurs and Mura (1964), and Dundurs (1969) for edge and screw dislocations near a perfect bimaterial interface in the limit of infinitely stiff undislocated material. We assume that a dislocation is created by slip discontinuity along the positive x_1 axis.

B.1 Edge dislocation with a Burgers vector b_1

The Airy stress function is

$$\Phi = -kb_1 \left[2(1-\nu) r \ln r \sin\theta + (1-2\nu) r\theta \cos\theta\right], \quad k = \frac{2\mu}{\pi(3-4\nu)}, \tag{B.1}$$

with the corresponding stresses

$$\sigma_r = -2\nu k b_1 \frac{\sin\theta}{r}, \quad \sigma_\theta = -2(1-\nu) k b_1 \frac{\sin\theta}{r}, \quad \sigma_{r\theta} = 2(1-\nu) k b_1 \frac{\cos\theta}{r}. \tag{B.2}$$

The displacement components are

$$u_r = \frac{b_1}{\pi} \begin{cases} (\theta - \pi/2)\cos\theta, & 0 < \theta \leq \pi/2, \\ (\theta + \pi/2)\cos\theta, & -\pi/2 \leq \theta < 0, \end{cases} \tag{B.3}$$

$$u_\theta = -\frac{b_1}{\pi} \begin{cases} (\theta - \pi/2)\sin\theta + \dfrac{\cos\theta}{3-4\nu}, & 0 < \theta \leq \pi/2, \\ (\theta + \pi/2)\sin\theta + \dfrac{\cos\theta}{3-4\nu}, & -\pi/2 \leq \theta < 0. \end{cases} \tag{B.4}$$

The Cartesian component counterparts are

$$u_1 = \frac{b_1}{\pi} \begin{cases} \theta - \pi/2 + \dfrac{\sin 2\theta}{2(3-4\nu)}, & x_2 > 0 \\ \theta + \pi/2 + \dfrac{\sin 2\theta}{2(3-4\nu)}, & x_2 < 0 \end{cases}, \qquad u_2 = -\frac{b_1}{\pi} \frac{\cos^2\theta}{3-4\nu}. \tag{B.5}$$

The latter are listed to point out that both displacement components (u_1 and u_2) are independent of the radial distance r. This independence of r is in contrast to the solution for the dislocation in an infinite medium, where u_2 component depends on r in a logarithmic manner.

The elastic strain energy within a large semicircle of radius R, outside a semicircular core of radius ρ, is

$$U = \frac{1}{2}\int_{\rho}^{R} (\sigma_{r\theta})_{\theta=0}\, b_1\, \mathrm{d}r + W_\rho + W_R\,, \quad W_\rho = -W_R\,, \tag{B.6}$$

where

$$\begin{aligned} W_\rho &= -\frac{1}{2}\int_{-\pi/2}^{\pi/2} (\sigma_r u_r + \sigma_{r\theta} u_\theta)_{r=\rho}\, \rho\, \mathrm{d}\theta = -\frac{\mu(1-2\nu)b_1^2}{2\pi(3-4\nu)^2}\,,\\ W_R &= \frac{1}{2}\int_{-\pi/2}^{\pi/2} (\sigma_r u_r + \sigma_{r\theta} u_\theta)_{r=R}\, R\, \mathrm{d}\theta = \frac{\mu(1-2\nu)b_1^2}{2\pi(3-4\nu)^2}\,. \end{aligned} \tag{B.7}$$

Since $(\sigma_{r\theta})_{\theta=0} = 2k(1-\nu)b_1/r$, there follows

$$U = E^{1|2} \ln\frac{R}{\rho}\,, \quad E^{1|2} = k(1-\nu)b_1^2\,, \tag{B.8}$$

where $E^{1|2}$ is the corresponding prelogarithmic factor.

B.2 Edge dislocation with a Burgers vector b_2

The Airy stress function is

$$\Phi = kb_2\left[2(1-\nu)r\ln r\cos\theta - (1-2\nu)r\theta\sin\theta\right], \tag{B.9}$$

with the corresponding stresses

$$\sigma_r = 2\nu k b_2 \frac{\cos\theta}{r}\,, \quad \sigma_\theta = 2(1-\nu)kb_2\frac{\cos\theta}{r}\,, \quad \sigma_{r\theta} = 2(1-\nu)kb_2\frac{\sin\theta}{r}\,. \tag{B.10}$$

The displacement components are

$$u_r = \frac{b_2}{\pi}\begin{cases} (\theta-\pi/2)\sin\theta - \dfrac{\cos\theta}{3-4\nu}\,, & 0<\theta\le\pi/2\,,\\ (\theta+\pi/2)\sin\theta - \dfrac{\cos\theta}{3-4\nu}\,, & -\pi/2\le\theta<0\,, \end{cases} \tag{B.11}$$

$$u_\theta = \frac{b_2}{\pi}\begin{cases} (\theta-\pi/2)\cos\theta\,, & 0<\theta\le\pi/2\,,\\ (\theta+\pi/2)\cos\theta\,, & -\pi/2\le\theta<0\,, \end{cases} \tag{B.12}$$

with the Cartesian counterparts

$$u_1 = -\frac{b_2}{\pi}\frac{\cos^2\theta}{3-4\nu}, \qquad u_2 = \frac{b_2}{\pi}\begin{cases}\theta - \pi/2 - \dfrac{\sin 2\theta}{2(3-4\nu)}, & x_2 > 0\\ \theta + \pi/2 - \dfrac{\sin 2\theta}{2(3-4\nu)}, & x_2 < 0\end{cases}. \tag{B.13}$$

The elastic strain energy is

$$U = \frac{1}{2}\int_\rho^R (\sigma_\theta)_{\theta=0}\, b_2\, \mathrm{d}r + W_\rho + W_R\,, \tag{B.14}$$

where

$$W_\rho = -W_R = \frac{\mu(3-2\nu)b_2^2}{2\pi(3-4\nu)^2}\,. \tag{B.15}$$

Since $(\sigma_\theta)_{\theta=0} = 2k(1-\nu)b_2/r$, there follows

$$U = E^{1|2}\ln\frac{R}{\rho}, \quad E^{1|2} = k(1-\nu)b_2^2\,, \tag{B.16}$$

where $E^{1|2}$ is the corresponding prelogarithmic factor.

B.3 Screw dislocation with a Burgers vector b_3

The displacement is

$$u_3 = \frac{b_3}{\pi}\begin{cases}\theta - \pi/2\,, & 0 < \theta \leq \pi/2\,,\\ \theta + \pi/2\,, & -\pi/2 \leq \theta < 0\,,\end{cases} \tag{B.17}$$

with the corresponding stresses

$$\sigma_{3r} = 0\,, \quad \sigma_{3\theta} = \frac{\mu b_3}{\pi r} = 2\sigma_{3\theta}^\infty\,. \tag{B.18}$$

The elastic strain energy is

$$U = \frac{1}{2}\int_\rho^R (\sigma_{3\theta})_{\theta=0}\, b_3\, \mathrm{d}r = E^{1|2}\ln\frac{R}{\rho}, \quad E^{1|2} = \frac{\mu b_3^2}{2\pi} = 2E_{b_3}^\infty\,. \tag{B.19}$$

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