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SPECTRAL STABILITY, STATIC AND OSCILLATORY INSTABILITY OF CIRCULATORY SYSTEMS

Abstract

The stability of linear mechanical systems subjected to potential and non-conservative positional forces is considered. A criterion which contains necessary and sufficient conditions for spectral stability, static (divergence) and oscillatory (flutter) instability of the systems is formulated. Particularly, this criterion directly implies a number of recent conditions for flutter in terms of the invariants of the system matrices.

SPEKTRALNA STABILNOST, STATIČKA I OSCILATORNA NESTABILNOST CIRKULATORNIH SISTEMA

Sažetak

Razmatra se stabilnost linearnih mehaničkih sistema podvrgnutih dejstvu potencijalnih i nepotencijalnih položajnih sila. Formulisan je kriterijum koji sadrži neophodne i dovoljne uslove spektralne stabilnosti, statičke nestabilnosti (divergencije) i oscilatorne nestabilnosti (flatera) datih sistema. Kao posljedica, iz ovog kriterijuma dobijeno je nekoliko nedavnih uslova oscilatorne nestabilnosti iskazanih preko invarijanti opisnih matrica sistema.

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INTRODUCTION

Non-conservative undamped linear systems (circulatory systems) are mostly expressed in the form

$$\ddot{q} + Kq + Cq = 0, \tag{1}$$

where dot denotes time differentiation, q is a n-dimensional vector, and real matrices $K = K^T$ and $C = -C^T$ are related to potential and non-conservative positional (circulatory) forces, respectively. Such systems are important mathematical models in areas of physics and engineering (solid mechanics, fluid dynamics, etc.). Tree classical examples of the systems are shown in Fig. 1–3 (for details, see [1–3]).

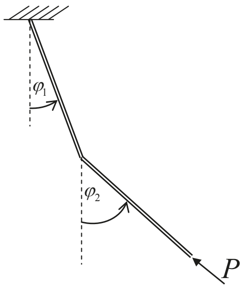

Fig. 1. Double pendulum with follower force

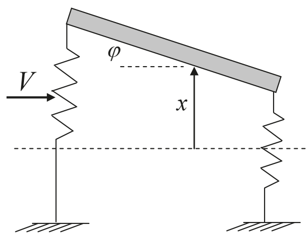

Fig. 2. Panel subjected to an airflow

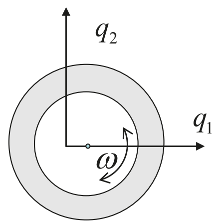

Fig. 3. Rotating shaft

The stability of system (1) is characterized by the position of the eigenvalues λ in the complex plane, where λ are the roots of the characteristic polynomial

$$\Delta(\lambda) = \det(\lambda^2 I + K + C) = \lambda^{2n} + a_1\lambda^{2(n-1)} + \ldots + a_{n-1}\lambda^2 + a_n \tag{2}$$

Since $\Delta(\lambda) = \Delta(-\lambda)$, then all eigenvalues are symmetric with respect to both real and imaginary axes (Fig.4).

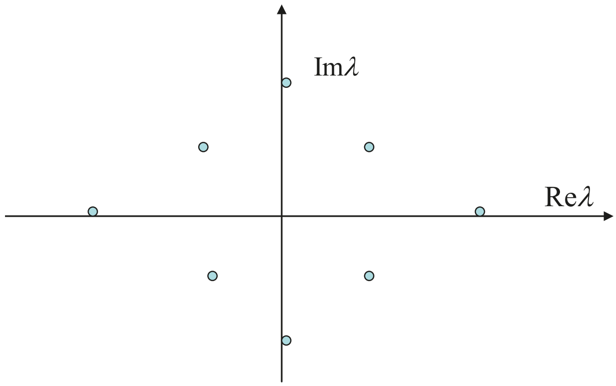

Fig. 4. Spectrum of the system

Definitions:

– The system (1) is *Lyapunov stable* if all eigenvalues λ lie on the imaginary axis and are semi-simple (algebraic and geometric multiplicities of λ coincide).

– The system (1) is *spectrally stable* if all eigenvalues λ lie on the imaginary axis.

– The system (1) is *statically unstable (divergence)* if at least one of λ is real while remaining eigenvalues belong to the imaginary axis.

– The system (1) is *oscillatory unstable (flutter)* if at least one of the eigenvalues λ is complex.

In general, Lyapunov stability implies spectral stability, but not conversely. Also, note that for systems (1) the boundaries of Lyapunov and spectral stability are identical. Consequently, the notion of spectral stability allows us to calculate stability limits without excluding multiple eigenvalue cases.

For almost a century it has been well known that circulatory forces can destabilize a stable potential system, and that they can stabilize an unstable potential system [1–4]. Various results concerning the stability problem for circulatory systems can be found in [1–9]. In what follows we give a criterion which contains necessary and sufficient conditions for spectral stability, static and oscillatory instability of the systems. The criterion is formulated in terms of the definiteness of a quadratic form whose coefficients are traces of powers of the system matrix (K+C).

A STABILITY CRITERION

Define the quadratic form

$$p(x) = x^T Px, \quad x \in \Re^n \tag{3}$$

with

$$P = \begin{pmatrix} n & p_1 & p_2 & \cdots & p_{n-1} \\ p_1 & p_2 & p_3 & \cdots & p_n \\ p_2 & p_3 & p_4 & \cdots & p_{n+1} \\ . & . & . & . & . \\ p_{n-1} & p_n & p_{n+1} & . & p_{2n-2} \end{pmatrix}, \tag{4}$$

where

$$p_k = (-1)^k TrA^k, \quad A = K + C, \quad k = 1,...,2n-2. \tag{5}$$

Theorem 1.

a) The system (1) is spectrally stable iff $p(x) \geq 0$ and $a_i \geq 0, i = 1,..., n$.
b) The system (1) is unstable by divergence iff $p(x) \geq 0$ and $\exists a_i < 0$.
c) The system (1) is unstable by flutter iff $p(x)$ can take negative values.

Remark 1. The coefficients a_i of the polynomial (2) can be expressed in terms of p_i by means of the Leverrier algorithm [9]

$$ka_k = -p_k - a_1 p_{k-1} - ... - a_{k-1} p_1, \quad k = 1,..., n. \tag{6}$$

Remark 2. This criterion is an alternative to the Gallina result [6] which requires the knowledge of the coefficients in the characteristic polynomial and the inspection of the minors of the 2n x 2n discriminant matrix.

Proof. $Tr(-A)^k = \sum_{i=1}^{n} \alpha_i^k$, k = 1, 2, … , where α_i are eigenvalues of $(-A)$ [10]. Then, in view of Borhardt-Jacobi theorem [10–11], rank and signature of the quadratic form (3) are equal to the number of unequal eigenvalues of $(-A)$ and the number of unequal real eigenvalues, respectively. It follows that all eigenvalues α_i are real iff $p(x)$ is positive semi-definite. In this case a simple consideration shows that a) all α are non-positive, i.e. all eigenvalues λ of the system (1) lie on the imaginary axis, iff $a_i \geq 0,\ i = 1,...,n$; b) at least one of the eigenvalues α is positive, i.e. at least one of λ is positive, iff $\exists a_i < 0$. Finally, in the case when $p(x)$ can take negative values at least one pair of the eigenvalues α is complex (at least two eigenvalues of λ are non-real and have positive real parts), and conversely.

SOME CONSEQUENCES

Lemma 1. If any of the inequalities hold

$$np_2 - p_1^2 < 0, \tag{7}$$

$$np_4 - p_2^2 < 0, \tag{8}$$

$$p_2 p_4 - p_3^2 < 0, \tag{9}$$

then $p(x)$ is not positive semi-definite.

Proof. The inequalities (7), (8) and (9) imply that the restrictions $p\,|_{x3=...=xn=0}$, $p\,|_{x2=x4=...=xn=0}$ and $p\,|_{x1=x4=...=xn=0}$ can take negative values, respectively.

From (5) by a direct computation we have

$$p_1 = -Tr(K), p_2 = \| K \|^2 - \| C \|^2, p_3 = -Tr(K^3) - 3Tr(KC^2), \tag{10}$$

$$p_4 = || K^2 ||^2 + || C^2 ||^2 - 4 || KC ||^2 + 2Tr((KC)^2), \tag{11}$$

where $\| B \|$ is Euclidean norm of a matrix B, i. e. $|| B ||^2 = \sum_{i,j} b_{ij}^2$.

Substituting expressions (10) and (11) in Lemma 1, according to Theorem1-c, we obtain the following recent instability results [8],[9].

Theorem 2. The system (1) is unstable by flutter if one of the following inequalities hold

a) $n(|| K ||^2 - || C ||^2) < Tr^2(K)$;[8] (12)

b) $n(|| K^2 ||^2 + || C^2 ||^2 - 4 ||KC||^2 + 2Tr((KC)^2)) < (|| K ||^2 - || C ||^2)^2$;[9] (13)

c) $(|| K ||^2 - || C ||^2)(|| K^2 ||^2 + || C^2 ||^2 - 4 || KC ||^2 + 2Tr((KC)^2)) < (Tr(K^3) + 3Tr(KC^2))^2$;[9] (14)

AN ILLUSTRATIVE EXAMPLE

Consider the tree degree of freedom system with

$$K=\begin{pmatrix}1&0&0\\0&1&k\\0&k&1\end{pmatrix},\quad C=c\begin{pmatrix}0&1&0\\-1&0&0\\0&0&0\end{pmatrix},\tag{15 }$$

where k and c are real numbers.

From (10) and (11), we have

$$p_1=-3\,,\ p_2=3+2(k^2-c^2),$$
$$p_4=3+12(k^2-c^2)+2(k^2-c^2)^2,\tag{16}$$

and

$$a_1=3,\ a_2=3+c^2-k^2,\ a_3=1+c^2-k^2.\tag{17}$$

In this case the leading principal minors of the matrix (2) are as follows

$$P_{(1,1)}=3,\ P_{(1,2)}=6(k^2-c^2),\ P_{(1,2,3)}=4(k^2-c^2)^3,\tag{18}$$

and, according to Theorem 1, the system is: spectrally stable iff $0\le k^2-c^2\le 1$, unstable by divergence iff $k^2-c^2>1$ and unstable by flutter iff $k^2-c^2<0$, see Fig.5.

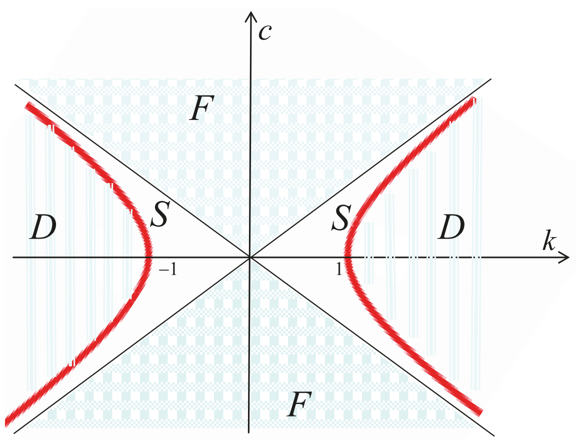

Fig. 5. Stability (S), divergence (D) and flutter (F) domains for the example

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