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Schwarz lemma and optimal recovery of functions in H^2

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Let $D \subset C^k$ be a domain, ν be a probability measure on $\overline{D}$ and X be a closed subspace of $L^2(\nu)$. Consider $D_0, \dots, D_n \subset D$ and probability measures $\mu_0, \dots, \mu_n$ on $D_0, \dots, D_n$ respectively. We suppose that $X \subset L^2(\mu_j), j = 0, 1, \dots, n$. We allow one of D_j to coincide with D. In this case we assume that μ_j coincides with ν.

Write $\mathcal{D} = (D_0, \dots, Dn), \mu = (\mu_0, \dots, \mu_n), \mu = (\mu_1, \dots, \mu_n), y = (y_1, \dots, y_n)$.

1. Optimal recovery problem

Given $y_1, \dots, y_n$ defined on $D_1, \dots, D_n$ such that

$$\|f_j - y_j\|_{L^2(\mu j)} \leq \delta_j,\ j = 1, \dots, n,$$

we are to reconstruct f. Here f_j is the restriction of f to D_j and $delta_j \geq 0, j = 1, \dots, n$ are accuracy levels. In particular, $delta_j = 0$ means that f is known precisely on D_j

A recovery algorithm (method, procedure, etc.) is an operator

$$A : L^2(\mu_1) \times \cdots \times L^2(\mu_n) \mapsto L^2(\mu_0).$$

We consider $A(y), y = (y_1, \dots, y_n)$, to be the recovered value of f on D_0. At this point we impose no conditions on A.

The maximal possible error of a method A is

$$e(X,\mathcal{D},\mu,\delta,A)=\sup\{\|f_0-A(y)\|_{L^2(\mu_0)}:f\in X, y\in L^2(\mu_1)\times\cdots\times L^2(\mu_n),\\ \|f_j-y_j\|_{L^2(\mu_j)}\le\delta_j, j=1,\dots,n\}$$

The optimal recovery error as

$$E(X,\mathcal{D},\mu,\delta)=\inf_{A:L^2(\mu_1)\times\cdots\times L^2(\mu_n)\mapsto L^2(\mu_0)}e(X,\mathcal{D},\mu,\delta,A).$$

A method $\widehat{A}$ such that

$$E(X,\mathcal{D},\mu,\delta)=e(X,\mathcal{D},\mu,\delta,\widehat{A})$$

is called an *optimal recovery method.*

The problem of finding an optimal recovery method (and sometimes an extremal function at which the optimal recovery error is attained) is usually referred to as *optimal recovery problem.*

2. Extremal problem

The optimal recovery problem is closely related to the following extremal problem. Find

$$\|f_0\|_{L^2(\mu 0)}\mapsto\sup,\ f\in X,\ \|f_j\|^2_{L^2(\mu_j)}\le\delta_j^2,\ j=1,\dots n. \tag{1}$$

A special case of this extremal problem is when D is the unit disk $\mathbb{D}$, μ_0 and μ_1 are point masses and μ_2 is the normalized Lebesgue measure on the unit circle. Here the problem turns into

$$\max\{|f(a_0)|:|f(a_1)|\le\delta_1,\ \|f\|_{H^2}\le\delta_2\},$$

which might be viewed as a version of the classical Schwarz lemma. Here we consider another variant of Scwarz Lemma. Let $a\in\mathbb{D}$ and Γ be a circle inside of the unit disk, μ be the normalized Lebesgue measure on Γ, and $\mu>0$. Find

$$\sup\left\{\int_\Gamma|f|^2d\mu : f\in H^2,\ \|f\|^2_{H^2}\le 1,\ |f(a)|\le\delta\right\}. \tag{2}$$

We will consider the case when the circle Γ passes through the origin and its center lies on the real axis, so that

$$\Gamma=\{z\in\mathbb{C}:|z-\rho|=\rho\},\ \ 0<\rho<1/2.$$

The corresponding optimal recovery problem is: *Reconstruct a Hardy function f from its values on the circle Γ and at a given with some tolerance.*

3. Euler equation for the general problem

Let $K(z,w)$ be the reproducing kernel of X. Write

$$\tilde{\mu} = -\mu_0 + \sum_{j=1}^{n} \lambda_j \mu_j.$$

Then $\tilde{\mu}$ is a regular measure on D and every function from X is square- integrable with respect to $\tilde{\mu}$. For $w \in D$ we introduce

$$d\tilde{\mu}_w(z) = K(z,w) d\tilde{\mu}(z).$$

Obviously every function from X is $\tilde{\mu}_w$-integrable.

We further define

$$\tau_w^{\lambda}(z) = \int_D K(z\tau) d\tilde{\mu}_w(\tau).$$

Theorem 1. *If $\widehat{f} \in X$ is a solution of the general extremal problem above, then there exists a non-negative vector $\widehat{\lambda} = (\widehat{\lambda}_1, \ldots, \widehat{\lambda}_n)$ such that*

$$\widehat{f} = (span\{\tau_w^{\widehat{\lambda}}, w \in D\})^{\perp},$$

and

$$\widehat{\lambda}_j(\|f\|_{L_2(\mu_j)} - \delta_j) = 0,\ j = 1, \ldots, n.$$

We say that a non-negative vector $\lambda = (\lambda_1, \ldots, \lambda_n)$ belongs to the spectrum of the problem, if there exists an admissible for this problem function $f \in X$ such that

1. $\lambda_j(\|f\|_{L^2(\mu_j)} - \mu_j) = 0.$
2. $f \in (spann\{\tau_w^{\lambda} : w \in D\})^{\perp}.$

In this case we call f a *spectral function.*

Theorem 2. *Let Λ be the spectrum of the problem. Then*

$$\sup_{\|f\|_{L_2(\mu_j)} \leq \delta_j, j=1,\ldots,n} = \sup_{\lambda \in \Lambda} \sum_{j=1}^{n} \lambda_j \delta_j^2. \tag{3}$$

We call a spectral point $(\widehat{\lambda_1}, \ldots, \widehat{\lambda_n})$ *extremal,* if the maximum of the right-hand side of (3) is attained at $(\widehat{\lambda_1}, \ldots, \widehat{\lambda_n})$.

4. Spectrum of the Schwarz Lemma

Here we have:

$$\tau_w^\lambda = -\frac{1}{\pi}\int_\Gamma \frac{1}{1-z\overline{\tau}}\cdot\frac{1}{1-\tau\overline{w}}\cdot\frac{|d\tau|}{|\tau-\rho|}+$$

$$\lambda_1\frac{1}{1-z\overline{a}}\cdot\frac{1}{1-a\overline{w}}+\frac{\lambda_2}{2\pi}\int_{|\tau|=1}\frac{1}{1-z\overline{\tau}}\cdot\frac{1}{1-\tau\overline{w}}|d\tau| =$$

$$-\frac{1}{1-z\rho-\rho\overline{w}}+\frac{\lambda_1}{(1-z\overline{a})(1-a\overline{w})}+\frac{\lambda_2}{1-z\overline{w}}.$$

By Theorem 1 every extremal function satisfies the following equation

$$\frac{1}{1-\rho w}f\left(\frac{=\rho}{1-\rho w}\right)=\lambda_1\frac{f(a)}{1-\overline{a}w}$$

for some $\lambda_1, \lambda_2 \geq 0$ and all $w \in \mathbb{D}$. Let

$$b=\frac{1-\sqrt{1-4\rho^2}}{2\rho}.$$

Then b is the Denjoy-Wolff point of the following self-mapping of $\mathbb{D}$

$$z \mapsto \frac{\rho}{1-\rho z},$$

and the disk bounded by the circle Γ is a hyperbolic neighborhood of b.

Consider the following functions

$$\varphi_j(z)=\frac{\sqrt{1-b^2}}{1-bz}\left(\frac{b-z}{1-bz}\right)^j,\ j=0,1,\ldots.$$

These functions form an orthonormal basis of H^2, and they are eigenfunctions of the operator

$$Tf(z)=\frac{1}{1-\rho z}f\left(\frac{\rho}{1-\rho z}\right),$$

and the corresponding eigenvalues are

$$\alpha_j=\frac{b^{2j}}{1-\rho b}. \tag{4}$$

Theorem 3. *Let $a \neq b$.*

1. *If*

$$\left| a - \frac{\rho}{1-\rho^2} \right| \geq \frac{\rho^2}{1-\rho^2},$$

or

$$\delta > \frac{\sqrt{|a|^2\rho^2 - |\rho - a|^2}}{a\rho + \overline{a}\rho - |a|^2},$$

then the spectrum of Schwarz Lemma extremal problem consists of two parts $\Lambda = \Lambda_1 \cup \Lambda_2$, where

$$\Lambda_1 = \{(0, \alpha_j) : |\varphi_j(a)| \leq \delta\},$$

$$\Lambda_2 = f(\lambda_1, \lambda_2) : \lambda_1, \lambda_2 > 0,$$

$$F(\lambda_2) = \delta^{-2},\ \lambda_1 = h(\lambda_2)$$

where

$$F(\lambda) = \sum_{j=0}^{\infty} \frac{|\varphi_j(a)|^2}{(a_j - \lambda)^2} h^2(\lambda),$$

$$h(\lambda) = \left(\sum_{j=0}^{\infty} \frac{|\varphi_j(a)|^2}{a_j - \lambda} \right)^{-1}.$$

2. *If*

$$\left| a - \frac{\rho}{1-\rho^2} \right| < \frac{\rho^2}{1-\rho^2},$$

and

$$\delta \leq \frac{\sqrt{|a|^2\rho^2 - |\rho - a|^2}}{a\rho + \overline{a}\rho - |a|^2},$$

then the spectrum includes in addition the point

$$\Lambda_3 = \left\{ \left(\frac{a\rho + \overline{a}\rho - |a|^2}{\rho^2}, 0 \right) \right\}.$$

Theorem 4. *Let $a = b$,*

$$\Lambda_1 = \{(0, \alpha_j) :\ j = 1, 2, \ldots, \},$$

$$\Lambda_2 = \{((1 - b^2)(\alpha_0 - \alpha_j), \alpha_j) :\ j = 1, 2, \ldots, \}.$$

Then the spectrum of problem is $\Lambda = \Lambda_1 \cup \Lambda_2$*, if* $\delta < \frac{1}{\sqrt{1-b^2}}$*, and* $\Lambda = \Lambda_1 \cup \Lambda_2 \cup \{(0, \alpha_0)\}$*, if* $\delta \geq \frac{1}{\sqrt{1-b^2}}$.

It turns out that Λ_2 is the most important part of the spectrum.

Proposition 1. *If a lies outside Γ, then $F(\lambda) \to \infty$ as $\lambda \to 0$.*

This Proposition implies that if a lies outside Γ, then Λ_2 contains only finite number of points.

Now we will use Theorem 2 to describe the extremal points of the spectrum.

Proposition 2. *If $\delta \geq |\varphi_0(a)|$, then $(0, \alpha_0)$ is the extremal point of the spectrum.*

Proposition 3. *If $a = b$ and $\delta < 1/\sqrt{1-b^2}$, then the extremal spectral point is*

$$(\widehat{\lambda}_1, \widehat{\lambda}_2) = ((1-b^2)(\alpha_0 - \alpha_1), \alpha_1).$$

Proposition 4. *If $\delta < |\varphi_0(a)|$, then Λ_1 does not contain extremal spectral points.*

Note that the function

$$g(\lambda) = \sum_{j=0}^{\infty} \frac{|\varphi_j(a)|^2}{\alpha_j - \lambda} \tag{7}$$

is monotone and increases from $-\infty$ to $+\infty$ when $\lambda \in (\alpha_{j+1}, \alpha_j)$. Let ζ_j be the only zero of g on the interval (α_{j+1}, α_j).

Proposition 5. *Let $a \neq b$. If $\delta \leq |\varphi_1(a)|$, then the extremal spectral point $(\widehat{\lambda}_1, \widehat{\lambda}_2)$ is unique, belongs to Λ_2 and is determined by the condition $\zeta_0 < \widehat{\lambda}_2 < \alpha_0$.*

Proposition 6. *Assume that $|\varphi_1(a)| < \delta < |\varphi_0(a)|$ and*

$$\gamma = \left| \frac{b-a}{1-ab} \right| \geq b^{2/3},$$

then the conclusion of Proposition 5 is valid, that is, the extremal spectral point $(\widehat{\lambda}_1, \widehat{\lambda}_2)$ is unique, belongs to Λ_2 and is determined by the condition that $\zeta_0 < \widehat{\lambda}_2 < \alpha_0$.

5. Optimal Recovery Method

The following result is due to Magaril-Il'yaev and Osipenko.

Theorem 5. *Assume that there exist* $\widehat{\lambda}_j \geq 0$, $j = 1, \dots, n$, *such that the value of the extremal problem*

$$\|f_0\|^2_{L_2(\mu_0)} \mapsto \max,$$

$$\sum_{j=1}^{\infty} \widehat{\lambda}_j \|f_j\|^2_{L_2(0,\mu_j)} \leq \sum_{j=1}^{\infty} \widehat{\lambda}_j \delta_j^2, \; f \in X$$

is the same as in (1). Moreover, assume that for every $\tilde{y} = (\tilde{y}_1, \dots, \tilde{y}_n) \in Y_1 \times \dots \times Y_n$, *where* Y_j *are almost everywhere dense in* $L^2(\mu_j)$, *there exists* $f_{\tilde{y}}$ *which is a solution of the extremal problem*

$$\sum_{j=1}^{\infty} \widehat{\lambda}_j \|f_j - \tilde{y}_j\|^2_{L_2(0,\mu_j)} \to \min, \; f \in X. \tag{8}$$

Moreover, let $\widehat{A} : L^2(\mu 1) \times \dots \times L^2(\mu_n) \mapsto L^2(\mu_0)$ *be a linear continuous operator, where the norm in* $L^2(\mu 1) \times \dots \times L^2(\mu_n)$ *is defined as*

$$\|y\| = \left(\sum_{j=1}^{n} \|y_j\|_{L^2(\mu_j)} \right)^{1/2},$$

such that for all $\tilde{y} = (\tilde{y}_1, \dots, \tilde{y}_n) \in Y_1 \times \dots \times Y_n$,

$$\widehat{A}(\tilde{y}) = (f_{\tilde{y}})_0.$$

Then

$$E(X, \mathcal{D}, \mu, \delta) = \sup_{\substack{f \in X \\ \|f_j\|_{L_2(\mu_j)} \leq \delta_j, \, j=1,\dots,n}} \|f_0\|_{L^2(\mu_0)}$$

and the method $\widehat{A}(y)$ *is optimal.*

We will apply Theorem 5 to the construction of optimal recovery method for the Schwarz Lemma type problem considered above.

Consider the extremal problem

$$\int_{\Gamma} |f|^2 d\mu \to \sup, \tag{9}$$

$$f \in H^2,\ \widehat{\lambda}_1|f(a)|^2 + \widehat{\lambda}_2\|f\|^2_{H^2} \le \widehat{\lambda}_1\delta^2 + \widehat{\lambda}_2$$

where as before μ is the normalized Lebesgue measure on $(\widehat{\lambda}_1, \lambda_2)$ is an extremal spectral point for problem (2).

Proposition 7. *Suppose that either*
1. $a \neq b$ *and* $\delta \le |\varphi_1(a)|$, *or* $|\varphi_1(a)| < \delta < |\varphi_0(a)|$ *and* $\gamma = \left|\frac{b-a}{1-ab}\right| \ge b^{2/3}$, *or*
2. $a = b$ *and* $\delta < \varphi(b) = 1/\sqrt{1-b^2}$.

Then the values of extremal problems (2) and (9) are the same.

Theorem 6. *Suppose that one of the following conditions is satisfied*
1. $\delta \ge |\varphi_0(a)|$,
2. $\delta \le |\varphi_1(a)|$,
3. $|\varphi_1(a)| < \delta < |\varphi_0(a)|$, $\gamma \ge b^{2/3}$,
4. $a = b$,

and $(\widehat{\lambda}_1, \widehat{\lambda}_2)$ *the corresponding extremal spectral point. Then the error of optimal recovery is given by*

$$\sqrt{\widehat{\lambda}_1\delta^2 + \widehat{\lambda}_2}$$

and the method

$$\widehat{A}(y)(z) = \frac{\widehat{\lambda}_1 y}{\widehat{\lambda}_1 + \widehat{\lambda}_2(1-|a|^2)} \cdot \frac{1-|a|^2}{1-\overline{a}z} \tag{10}$$

is optimal.

Note that for $a = b$ the optimal method of recovery (10) does not depend on δ and has the form

$$\widehat{A}(y)(z) = \frac{1-|b|^2}{1-bz}.$$

6. Open problems

1. It would be desirable to identify the extremal spectral point in all possible cases. We have shown that in a number of cases the extremal spectral point is the only point in Λ_2 such that $\zeta_0 < \widehat{\lambda}_2 < \alpha_0$. Our attempts to find a nontrivial-case when this point is not extremal failed. Thus, we are tempted to conjecture that the point of Λ_2 with the biggest λ_2 is always extremal.

Conjecture. *If* $a \neq b$ *and* $\delta < |\varphi_0(a)|$, *the point in* Λ_2 *such that* $\zeta_0 < \widehat{\lambda}_2 < \alpha_0$. *is always the spectral extremal point for problem (2).*

2. It is natural to ask which choice of a minimizes the value of problem (2) (of course, this choice of a leads to the least optimal recovery error). It follows from above discussion that the point b plays a special role.

Problem. *Does the choice $a = b$ always lead to the least mean square optimal recovery error?*

3. Finally, if in problem (2) we replace the constraint $|\varphi(a)| \le \delta$ with

$$\frac{1}{2\pi i}\int_{|z-a|=r} |f(z)|^2 d(z-a) \le \delta,\ 0 < r < 1-|a|,$$

then the problem becomes even more difficult. The reason is that in the right hand side of Euler's equation the term $\lambda_1 \frac{f(a)}{1-\bar{a}z}$ is replaced with

$$\lambda_1 f\left(a - \frac{r^2 z}{1-\bar{a}z}\right).$$

and the equation turns into

$$\frac{1}{1-\rho w} f\left(\frac{\rho}{1-\rho w}\right) = \frac{\lambda_1}{1-\bar{a}z} f\left(a - \frac{r^2 z}{1-\bar{a}z}\right) + \lambda_2 f(w)$$

Thus, finding the spectrum in this case is reduced to finding eigenvalues of an operator which is a linear combination of two compact non-commuting operators. It would be very interesting to find the eigenbasis which corresponds to this problem and to find the solution.

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