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ON REGULARIZATION OF ONE QUADRATIC PROGRAMMING PROBLEM WITH APPROXIMATE INITIAL DATA

Abstract

In this paper we consider a problem of minimization of a quadratic function on an ellipsoid in a Hilbert space. In general case, this problem is ill-posed. This fact generates the necessity to apply certain method of regularization, [1], [3], [13] that will produce a good approximate solutions of the problems. Methods of regularization that will be used in this paper are based on a modification of the family of regularizing functions from [12]. We used this method in [6] for solving the same problem under assumptation that the ellipsoid is known exactly, while instead of all other data we know only their approxmations. Here, we assume that insted of the exact initial data, we know only their approximations.

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Key words: quadratic programming, ill-posed problems, regularization

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O REGULARIZACIJI JEDNOG ZADATKA KVADRATNOG PROGRAMIRANJA SA PRIBLIŽNIM POČETNIM PODACIMA

Apstrakt

U ovom radu izučavamo zadatak minimizacije kvadratnog funkcionala na elipsoidu u Hilbertovom prostoru. U opštem slučaju ovaj problem je nekorektan i za njegovo rješavanje potrebno je primijeniti metode regularizacije (v. [1], [3], [13]) koje će generisati dobre aproksimacije rješenja. Metode regularizacije koje će biti korišćene u ovom radu zasnovane su na modifikaciji klase regularizujućih funkcija iz [12]. U radu [6] razmatrali smo isti problem pretpostavljajući da su svi parametri kojima se definiše elipsoid poznati, dok su umjesto ostalih parametara poznate samo njihove aproksimacije. Ovdje pretpostavljamo da su umjesto tačnih početnih podataka poznate samo njihove aproksimacije.

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Ključne riječi: kvadratno programiranje, nekorektni zadaci, regularizacija

1 Introduction

Let H and F be real Hilbert spaces, $A : H \mapsto F$ - linear bounded operator from H to F, $f \in F$- a fixed element, and $U \subseteq H$- a closed convex set. We will deal with the minimization problem

$$J(u) = \frac{1}{2}\|Au - f\|^2,\ u \in U. \tag{1.1}$$

We suppose that the set U is given by

$$U = \{u \in H : \|Bu\|^2 \leq r^2\}, \tag{1.2}$$

where $B : H \mapsto G$ is an linear continuous operator from H to Hilbert space G.

Even when the constraint $u \in U$ is absent, this problem can be ill-posed, i.e. it is possible that there is $\tilde{u}$ which is far from the set of solution of (1.1), such that $\|A\tilde{u} - f\| \approx \min\{\|Au - f\|^2 : u \in U\}$. In this case, the ill-posedness of problem (1.1) with $U = H$, comes obviously from the fact that the range of A, $R(A) := \{Au : u \in H\}$ is not closed. In case of constraints, the ill-posednes can be a consequence of the properties of the operator B and of the structure of the set U_* of solutions of the problem [5].

Problem (1.1), (1.2), and the correspondng problem without constraints

$$J(u) = \frac{1}{2}\|Au - f\|^2,\ u \in H. \tag{1.3}$$

in literature (see [12], [4], [7], [8] are regularly studied under assumption that instead of the exact operators A and/or B and instead of the element f one actually deals with their approximations $A_\eta \in \mathcal{L}(H, F)$, $f_\delta \in F$, and $B_\sigma \in \mathcal{L}(H, G)$, such that

$$\|A - A_\eta\| \leq \eta,\ \ \|B - B_\sigma\| \leq \sigma,\ \|f - f_\delta\| \leq \delta, \tag{1.4}$$

where $\eta > 0, \sigma > 0$, and $\delta > 0$ are small positive real numbers.

Then, problem (1.1), (1.2) is ill-posed in many important concrete cases, and in order to solve it, one has to use methods of regularization. In this paper, we consider methods of regularization based on a modification of the family of regularizing functions from [12]. Let us mention that these methods were used in [6] in case when the operator B is known exactly. Note also that the Tikhonov method and the iterated Tikhonov method of regularization belong to this class of methods. Usually, the estimates of the accuracy of of the regularization methods for solving ill-posed problems (1.1), (1.2) are obtained for classes of the problems defined by certain conditions related to their soultons u_* with minimal norm. In general case, these so-called *source conditons* have the following form

$$u_* = \varphi(A^*A)h_*,\ h_* \in H$$

and were discussed, for example, in [2], [9], [10].

Problems of the (1.1) and (1.3) with infinite-dimensionale spaces H and/or F are usually related to optimal control problems (see for example see[13].)

2 Regularization Method

Let us suppose that the set U_* of solutons of problem (1.1), (1.2) is nonempty. We will denote by u^* and call a normal soluton of (1.1) the element of U_* with minimal norm: $u^* \in U_*$ and $\|u_*\| \leq \|u\|$ for all $u \in U_*$.

As an approximation of u^* of (1.1), (1.2), one can take a unique soluton $w_\alpha = w_{\alpha\eta\delta\sigma}$ of the variational inequality ([12] and [6]),

$$\left\langle g_\alpha^{-1}(A_\eta^* A_\eta)u_\alpha - A_\eta^* f_\delta, u - u_\alpha \right\rangle \geq 0, \ \forall u \in U_\sigma := \{u \in H : \|B_\sigma u\|^2 \leq r^2\}. \tag{2.1}$$

where Borel measurable functions $g_\alpha : [0, a] \mapsto \mathbb{R}$, $a > 0, \alpha > 0$ ($a \geq \|A_\eta\|$) satisfy the conditions:

$$1 - t g_\alpha(t) \geq 0, \, t \in [0, a], \tag{2.2}$$

$$\frac{1}{1 + \beta\alpha} \leq g_\alpha(t) \leq \frac{1}{\beta\alpha}, \, t \in [0, a], \ \beta > 0, \tag{2.3}$$

$$\sup_{0 \leq t \leq a} t^p (1 - t g_\alpha(t)) \leq \gamma_p \alpha^p, \, \alpha > 0, \, \gamma_p = const, \, 0 \leq p \leq p_0, \, p_0 > 0. \tag{2.4}$$

Here, a is the constant such thata $a \geq \|A_\eta$. Number p_0 is called *qualification of the family* $\{g_\alpha : \alpha > 0\}$ and it has an important role.

Let us denote by u_α a unique solution to (2.1) on the set $U = \{u \in H : \|Bu\|^2 \leq r^2\}$. In [6] (see also [4]) the following result was proven.

Teorema 1. *Suppose conditions (1.4) and (2.2)-(2.4) are satisfied.*

(a) If the parameter α in (2.1) is chosen such that $\alpha = \alpha(\eta, \delta) \to 0$ and $\frac{\eta + \delta^2}{\alpha} \to 0$ as $\eta, \delta \to 0$, then $u_\alpha \to u_*$ as $\eta, \delta \to 0$.

(b) If $u_* = |A|^p h_*$, where $h_* \in H$, $|A|^p = (A^* A)^{\frac{p}{2}}$, $p > 0$, and

$$\alpha = \alpha(\eta, \delta) = d(\eta + \delta)^{\frac{2}{p+2}}, \, d = const,$$

then

$$\|u_\alpha - u_*\| \leq const(\eta+\delta)^{\frac{p}{p+2}},\ 0 \leq p \leq 2p_0 - 1. \tag{2.5}$$

Let us estimate the distance between w_α and u_α.

Lemma 2.1. *There is a constant* $c_0 > 0$ *such that*

$$\|u_\alpha - w_\alpha\| \leq c_0 \frac{\sigma}{\alpha}.$$

Proof. By the Khun-Tucker theorem, there exist real numbers $\lambda_\alpha \geq 0$ and $p_\alpha \geq 0$, such that

$$g_\alpha^{-1}(A_\eta^* A_\eta)u_\alpha - A_\eta f_\delta + \lambda_\alpha B^* B u_\alpha = 0 \tag{2.6}$$

$$\lambda_\alpha \|Bu_\alpha\|^2 = \lambda_\alpha r^2 \tag{2.7}$$

$$g_\alpha^{-1}(A_\eta^* A_\eta)w_\alpha - A_\eta f_\delta + p_\alpha B_\sigma^* B_\sigma w_\alpha = 0 \tag{2.8}$$

$$p_\alpha \|B_\sigma w_\alpha\|^2 = p_\alpha r^2 \tag{2.9}$$

Firstly, let us prove the boundeness of the Lagrange multipliers λ_α and p_α from (2.6) - (2.9). Multiplying (2.6) by u_α, (2.8) by w_α and bearing in mind (2.7) and (2.9), we obtain

$$\|g_\alpha^{-1}(A_\eta^* A_\eta)u_\alpha\|^2 + \lambda_\alpha r^2 = \langle f_\delta, A_\eta u_\alpha\rangle, \tag{2.10}$$

and

$$\langle g_\alpha^{-1}(A_\eta^* A_\eta)w_\alpha, w_\alpha\rangle + p_\alpha r^2 = \langle f_\delta, A_\eta^* w_\alpha\rangle. \tag{2.11}$$

From here, taking into account the inequalty $\|A_\eta x\|^2 \leq \langle g_\alpha^{-1}(A_\eta^* A_\eta x, x\rangle$ for all x, we obtain that $\|A_\eta u_\alpha\| \leq \|f_\delta\|$ and $\|A_\eta w_\alpha\| \leq \|f_\delta\|$. Consequently,

$$max\{\lambda_\alpha; p_\alpha\} \leq \frac{\|f_\delta\|^2}{r^2} \leq \frac{2(\|f\|^2 + \delta^2)}{r^2}.$$

Further more, if $u_\alpha^* = g_\alpha(A_\eta^* A_\eta)f_\delta$ belongs to $\{u \in H : \|B\|^2 < r^2\}$, then $\|B_\sigma u_\alpha^*\| < r^2$ for all suffciently small σ and $u_\alpha = w_\alpha = u_\alpha^*$. This equality is valid also in case of $\|Bu_\alpha^*\| = r^2$ and$\|B_\sigma u_\alpha^*\| < r^2$. In all other cases we have that

$$\|Bu_\alpha\|^2 = r^2,\ \|B_\sigma w_\alpha\|^2 = r^2. \tag{2.12}$$

Hence, in what follows, we will assume that this is fullfiled.

Multiplying the equality

$$g_\alpha^{-1}(A_\eta^* A_\eta)u_\alpha - g_\alpha^{-1}(A_\eta^* A_\eta)w_\alpha = p_\alpha B_\sigma^* B_\sigma w_\alpha - \lambda_\alpha B^* B u_\alpha$$

by $u_\alpha - w_\alpha$, we have

$$\langle g_\alpha^{-1}(A_\eta^* A_\eta)(u_\alpha - w_\alpha), u_\alpha - w_\alpha\rangle \leq$$

$$p_\alpha\langle B_\sigma^* B_\sigma w_\alpha, u_\alpha - w_\alpha\rangle - \lambda_\alpha\langle B_\sigma^* B_\sigma u_\alpha, u_\alpha - w_\alpha\rangle + \lambda_\alpha\langle (B_\sigma^* B_\sigma - B^* B)u_\alpha)u_\alpha, u_\alpha - w_\alpha\rangle. \tag{2.13}$$

Now, from (2.13) using the following consequence of (3.5)

$$p_\alpha\langle B_\sigma^* B_\sigma w_\alpha, u_\alpha - w_\alpha\rangle - \lambda_\alpha\langle B^* B u_\alpha, u_\alpha - w_\alpha\rangle = -\frac{\lambda_\alpha + p_\alpha}{2}\|B_\sigma(u_\alpha - w_\alpha\|^2 \leq 0,$$

we obtain

$$\langle g_\alpha^{-1}(A_\eta^* A_\eta)(u_\alpha - w_\alpha), u_\alpha - w_\alpha\rangle \leq \lambda_\alpha\langle (B_\sigma^* B_\sigma - B^* B)u_\alpha, u_\alpha - w_\alpha\rangle =$$

$$\lambda_\alpha \left\langle (B_\sigma^*(B_\sigma - B) + (B_\sigma^* - B^*)B)\, u_\alpha, u_\alpha - w_\alpha\right\rangle.$$

Finally, taking into account the estimates

$$\langle g_\alpha^{-1}(A_\eta^* A_\eta)(u_\alpha - w_\alpha), u_\alpha - w_\alpha\rangle \geq \beta\alpha\|u_\alpha - w_\alpha\|^2 \text{ and } \lambda_\alpha \leq \frac{\|f_\delta\|^2}{r^2},$$

we obtain

$$\alpha\|u_\alpha - w_\alpha\|^2 \leq const \cdot \sigma.$$

This completes the proof of the Lemma.

3 Algorithm and Rate of Convergence

In what follows we will consider the convergence of the regularized approximations of the solution, which are obatined in real process, to u_*. At the begin, we will establish some properties of the functions $s_\alpha : [0, +\infty) toH$ and $\varphi : [0, +\infty) \to H$ definied by

$$s_\alpha : [0, +\infty) \mapsto H,\ s_\alpha(t) = \left(g_\alpha^{-1}(A_\eta^* A_\eta) + tB_\sigma^* B_\sigma\right)^{-1} A_\eta^* f_\delta,$$

$$\varphi_\alpha : [0,+\infty) \mapsto [0,+\infty),\ \psi_\alpha(t) = \|B_\sigma s_\alpha(t)\|^2.$$

Let us observe that the functions s_α satisfy the equality

$$g_\alpha^{-1}(A_\eta^* A_\eta)s_\alpha(t) + tB_\sigma^* B_\sigma s_\alpha(t) = A_\eta^* f_\delta. \tag{3.1}$$

This equation is obtained by applying of the operator $g_\alpha^{-1}(A_\eta^* A_\eta) + tB^*B$ on the equality $s_\alpha(t) = (g_\alpha^{-1}(A_\eta^* A_\eta) + tB_\sigma^* B_\sigma)^{-1} A_\eta^* f_\delta$.

Lemma 3.1. *(a) Functions s_α and φ_α are differentiable and*

$$s'_\alpha(t) = -(g_\alpha^{-1}(A_\eta^* A_\eta) + tB_\sigma^* B_\sigma)^{-1} B_\sigma^* B_\sigma s_\alpha(t),$$

$$\varphi'_\alpha(t) = -2\left\langle (g_\alpha^{-1}(A_\eta^* A_\eta) + tB_\sigma^* B_\sigma)^{-1} B_\sigma^* B_\sigma s_\alpha(t), B_\sigma^* B_\sigma s_\alpha(t)\right\rangle$$

(b) If the solution of the equation $g_\alpha^{-1}(A_\eta^* A_\eta)u - A_\eta^* f_\delta = 0$ does not belong to the set $U_\sigma = \{u : \|B_\sigma u\| \le r^2\}$, then the following statements hold:

(i) $B_\sigma^* B_\sigma s_\alpha(t) \neq 0$ for all $t \in [0,+\infty)$.

(ii) Function φ_α is strictly decreasing and $\lim\limits_{t\to+\infty} \varphi_\alpha(t) = 0$.

(iii) There exists $t_\alpha \in \left(0, \frac{\|f_\delta\|^2}{r^2-\sigma}\right)$ such that

$$r^2 - \sigma \le \|B_\sigma s_\alpha(t_\alpha)\|^2 \le r^2. \tag{3.2}$$

(iv) There exists $M > 0$ such that

$$\|\varphi'(t)| \ge M\ \forall t < t_\alpha. \tag{3.3}$$

Proof. By a simple trasformation, we obtain

$$\frac{s_\alpha(t+h) - s_\alpha(t)}{h} = -(g_\alpha^{-1}(A_\eta^* A_\eta) + (t+h)B_\sigma^* B_\sigma)^{-1} B_\sigma^* B_\sigma s_\alpha(t).$$

The first equality in (a) follows from here. Further more, it is easy to prove the second equality in (a), in the following way:

$$\varphi'_\alpha(t) = 2\langle s'_\alpha(t), B_\sigma^* B_\sigma s_\alpha(t)\rangle = -2\left\langle (g_\alpha^{-1}(A_\eta^* A_\eta) + tB_\sigma^* B_\sigma)^{-1} B_\sigma^* B_\sigma s_\alpha(t), B_\sigma^* B_\sigma s_\alpha(t)\right\rangle.$$

(b) Bu asumptation, $\|B_\sigma s_\alpha(0)\| > r^2$. Then, the solution u_α^* of the equation $g_\alpha^{-1}(A_\eta^* A_\eta)u - A_\eta^* f_\delta = 0$ is given by

$$u_\alpha^* = g_\alpha(A_\eta^* A_\eta)A_\eta^* f_\delta = s_\alpha(0).$$

(b1) If there is $t_0 > 0$ such that $B_\sigma^* B_\sigma s_\alpha(t_0) = 0$. then

$$s_\alpha(t_0) = g_\alpha(A_\eta^* A_\eta)A_\eta^* f_\delta = s_\alpha(0).$$

However, this is not possible, because we supposed that $B_\sigma^* B_\sigma s_\alpha(0) \neq 0$ and $B_\sigma^* B_\sigma s_\alpha(t_0) = 0$.

(b2) Operator $(g_\alpha^{-1}(A_\eta^* A_\eta) + tB_\sigma^* B_\sigma)^{-1}$ is positive and $B_\sigma^* B_\sigma s_\alpha(t) \neq 0$ for all $t \in [0, +\infty)$. Therefore, $\varphi'(t) < 0$ for all $t \in [0, +\infty)$ and, consequently, φ is strictly decreasing.

Multiplying scalarly equality (3.1) by $s_\alpha(t)$, we obtain

$$\left\langle g_\alpha^{-1}(A_\eta^* A_\eta)s_\alpha(t), s_\alpha(t)\right\rangle + t\|B_\sigma s_\alpha(t)\|^2 = \left\langle f_\delta, A_\eta s_\alpha(t)\right\rangle.$$

From here, using the inequality

$$\langle g_\alpha^{-1}(A_\eta^* A_\eta)s_\alpha(t), s_\alpha(t)\rangle \geq \|A_\eta s_\alpha(t)\|^2$$

we have

$$t\|B_\sigma s_\alpha(t)\|^2 = \langle f_\delta, A_\eta s_\alpha(t)\rangle - \left\langle g_\alpha^{-1}(A_\eta^* A_\eta)s_\alpha(t), s_\alpha(t)\right\rangle \leq$$

$$\frac{1}{2}\|A_\eta s_\alpha(t)\|^2 + \frac{1}{2}\|f_\delta\|^2 - \|A_\eta s_\alpha(t)\|^2$$

i.e. it yields

$$0 \leq \varphi(t) \leq \frac{\|f_\delta\|^2}{2t} - \frac{\|A_\eta s_\alpha(t)\|^2}{2t} \leq$$

$$\frac{\|f_\delta\|^2}{2t} \to 0 \text{ as } t \to \infty. \tag{3.4}$$

The Lemma is proven.

Consequently, if $\varphi(0) = \|Bs_\alpha(0)\| > r^2$, then there exists $t_0 > 0$ such that $\varphi(t_0) = r^2$. The monotonicity and the differentiability of the functions φ_α offer the possibility to apply different methods for approximate solving of the equation such that $\varphi_\alpha(t) = r^2$. Let us denote by $t_\alpha = t\alpha\varepsilon \approx t_0$ an approximate solution of this equation, such that $\|v_\alpha - w_\alpha\| = \|s_\alpha(t_\alpha - s(t_0)\| \leq \varepsilon$.

The following theorem characterizes the convergence of the described method.

Teorema 2. *Suppose that the conditions (1.4) and (2.2)-(2.4), and condition (b) from Theorem 1 are satisfied.*

(a) If the parameter $\alpha = \alpha(\eta, \delta, \sigma)$ *and* $t_\alpha > 0$ *are such that* $r^2 - (\eta + \delta + \sigma) < \|Bs_\alpha(t_\alpha)\|^2 \leq r^2$ *and* $\alpha(\eta, \delta) \to 0$, $\frac{\eta+\delta^2+\sigma}{\alpha} \to 0$ *as* $\eta, \delta, \sigma \to 0$, *then*

$$v_\alpha = \begin{cases} s_\alpha(0), & s_\alpha(0) \leq r^2 \\ s_\alpha(t_{\alpha)} & s_\alpha(0) > r^2, \end{cases}$$

converges to normal solution u_* *of problem (1.2), (1.1) as* $\eta, \delta\sigma, \varepsilon \to 0$.

(b) If $u_* = |A|^p h_*$, *where* $h_* \in H$, $|A|^p = (A^*A)^{\frac{p}{2}}$, $p > 0$, *and*

$$\alpha = \alpha(\eta, \delta, \sigma) = d(\eta + \delta)^{\frac{2}{p+2}}, \; d = const,$$

then

$$\|v_\alpha - u_*\| \leq const \left[(\eta + \delta)^{\frac{p}{p+2}} + \frac{\sigma}{\delta} + \varepsilon\right]., \; 0 \leq p \leq 2p_0 - 1. \qquad (3.5)$$

Proof. From the inequality

$$\|u_* - v_\alpha\| \leq \|u_* - u_\alpha\| + \|u_\alpha - w_\alpha\| + \|w_\alpha - v_\alpha\|. \qquad (3.6)$$

bearing in mind the results from Theorem 2.1, Lemma 2.1, we obtain the conclusions of the Theorem.

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