

ЦРНОГОРСКА АКАДЕМИЈА НАУКА И УМЈЕТНОСТИ  
ГЛАСНИК ОДЈЕЉЕЊА ПРИРОДНИХ НАУКА, 20, 2014.

ЧЕРНОГОРСКАЯ АКАДЕМИЯ НАУК И ИСКУССТВ  
ГЛАСНИК ОТДЕЛЕНИЯ ЕСТЕСТВЕННЫХ НАУК, 20, 2014.

THE MONTENEGRIN ACADEMY OF SCIENCES AND ARTS  
GLASNIK OF THE SECTION OF NATURAL SCIENCES, 20, 2014.

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UDK 514.764.27

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## **GEOMETRIC FORMALITY OF RATIONALLY ELLIPTIC MANIFOLDS IN SMALL DIMENSIONS**

### *A b s t r a c t*

We classify simply connected rationally elliptic manifolds of dimension five and those of dimension six with small Betti numbers from the point of view of their rational cohomology structure. We also prove that a geometrically formal rationally elliptic six-dimensional manifold, whose second Betti number is two, is rational cohomology  $S^2 \times \mathbb{C}P^2$ . An infinite family of six-dimensional simply connected biquotients whose second Betti number is three, different from Totaro's biquotients, is considered and it is proved that none of the biquotients from this family is geometrically formal.

MSC: 53C25, 53C30

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# GEOMETRIJSKA FORMALNOST RACIONALNO ELIPTIČKIH MNOGOSTRUKOSTI MALIH DIMENZIJA

*I z v o d*

U radu su, sa stanovišta racionalne kohomološke strukture, klasifikovane racionalno eliptičke mnogostrukosti u dimenziji pet i one u dimenziji šest sa malim Betijevim brojevima. Pokazano je da je geometrijski formalna racionalno eliptička šestodimenziona mnogostrukost čiji je drugi Betijev broj dva, racionalno kohomologna proizvodu  $S^2 \times \mathbb{C}P^2$ . Razmatrana je i beskonačna familija šestodimenzionih prosto povezanih bi-količnika čiji je drugi Betijev broj tri, a koji nijesu Tatarovi bi-količnici, i pokazano je da nijedan od ovih bi-količnika nije geometrijski formalna mnogostrukost.

## 1. INTRODUCTION

Geometric formality of a compact smooth manifold  $M$  is the notion introduced in [8] and it is concerned with an existence of a Riemannian metric on  $M$  such that the related harmonic forms form an algebra. Such a metric is called formal. Geometrically formal manifolds are formal in the sense of rational homotopy theory. Moreover, the original proof of the rational formality of symmetric spaces, which are one of the first non-trivial examples of such spaces, implicitly used the fact that the invariant metric on symmetric spaces is formal. But, it turns out that the notion of geometric formality is much more restrictive. It is proved in [8] that in dimension less or equal four, a geometrically formal manifold must have real cohomology of a symmetric space. Besides that, in [10] are provided a lot examples of homogeneous spaces which are rationally formal and which from cohomological reasons are not geometrically formal.

Therefore, the investigation of geometric formality in general, as well as for some family of spaces or particular examples remains unsolved, interesting and important problem for many applications in

pure mathematics and mathematical physics. In the recent time as well in the focus of mathematical interest is the connection between geometric formality property and different differential geometrical properties of manifolds such as sectional and scalar curvature [9], [1].

As noted, this problem should be studied in the category of rationally formal spaces. Besides that, the Hodge theory gives that for geometrically formal spaces the algebra structure of the harmonic forms is the same as its real cohomology algebra structure. It suggests that one can hardly expect the positive answer to the question of geometric formality for the manifolds with many cohomology relations. In this paper, from these reasons, we study rationally elliptic manifolds in the next two unsolved dimensions five and six. For these manifolds it is known to comprise simply connected homogeneous spaces and biquotients in corresponding dimensions.

## 2. GENERAL BACKGROUND

### 2.1. Rational homotopy theory

We refer to [6] for a comprehensive general reference for rational homotopy theory.

Let  $(\mathcal{A}, d_{\mathcal{A}})$  be a connected ( $H^0(\mathcal{A}, d_{\mathcal{A}}) = k$ ) and simply connected ( $H^1(\mathcal{A}, d_{\mathcal{A}}) = 0$ ) commutative  $\mathbb{N}$ -graded differential algebra over a field  $k$  of characteristic zero. Let us consider the free  $\mathbb{N}$ -graded commutative differential algebra  $(\wedge V, d)$  for a  $\mathbb{N}$ -graded vector space  $V$  over  $k$ . It is said that  $(\wedge V, d)$  is a minimal model for  $(\mathcal{A}, d_{\mathcal{A}})$  if  $d(V) \subset \wedge^{\geq 2} V$  and there exists a morphism

$$f : (\wedge V, d) \rightarrow (\mathcal{A}, d_{\mathcal{A}}) ,$$

which induces an isomorphism in cohomology.

Let  $X$  be a simply connected topological space of finite type. The minimal model  $\mu(X)$  for  $X$  is defined to be the minimal model for the algebra  $\mathcal{A}_{PL}(X)$  of piece-wise linear forms on  $X$ . One says that two simply connected manifolds have the same rational homotopy type

if and only if there is a third space to which they both map by maps inducing isomorphism in rational cohomology. Then the following facts are well known. The minimal model  $\mu(X)$  of a simply connected topological space  $X$  of finite type is unique up to isomorphism (which is well defined up to homotopy), it classifies the rational homotopy type of  $X$  and, furthermore, it contains complete information on the ranks of the homotopy groups of  $X$ . More precisely,

$$\text{rk } \pi_r(X) = \dim(\mu(X)/\mu^+(X) \cdot \mu^+(X))_r, \quad r \geq 2, \quad (2.1)$$

where by  $\mu^+(X)$  we denote the elements in  $\mu(X)$  of positive degree and  $\cdot$  is the usual product in  $\mu(X)$ . One of the equivalent definition of formality is that  $X$  is formal in the sense of Sullivan if its minimal model coincides with the minimal model of its cohomology algebra  $(H^*(X, \mathbb{Q}), d = 0)$  (up to isomorphism).

The procedure for minimal model construction is given, see [6], through the proof of the theorem which states the existence (and also the uniqueness up to isomorphism) of the minimal model for any such algebra. We briefly describe this procedure here, since we are going to apply it explicitly.

**2.1.1 Procedure for minimal model construction.** In the procedure for the construction of the minimal model for a simply connected commutative differential  $\mathbb{N}$ -graded algebra  $(\mathcal{A}, d)$  one starts by choosing  $\mu_2$  and  $m_2 : (\mu_2, 0) \rightarrow (\mathcal{A}, d)$  such that  $m_2^{(2)} : \mu_2 \rightarrow H^2(\mathcal{A}, d)$  is an isomorphism. In the inductive step, supposing that  $\mu_k$  and  $m_k : (\mu_k, d) \rightarrow (\mathcal{A}, d)$  are constructed we extend it to  $\mu_{k+1}$  and  $m_{k+1} : (\mu_{k+1}, d) \rightarrow (\mathcal{A}, d)$  with

$$\mu_{k+1} = \mu_k \otimes \mathcal{L}(u_i, v_j), \quad (2.2)$$

where  $\mathcal{L}(u_i, v_j)$  denotes the vector space spanned by the elements  $u_i$  and  $v_j$  corresponding to  $y_i$  and  $z_j$  respectively. The latter are given by

$$H^{k+1}(\mathcal{A}) = \mathfrak{S}m_k^{(k+1)} \oplus \mathcal{L}(y_i) \quad (2.3)$$

and

$$\text{Ker } m_k^{(k+2)} = \mathcal{L}(z_j). \quad (2.4)$$

Then we have that  $m_k(z_j) = dw_j$  for some  $w_j \in \mathcal{A}$  and the homomorphism  $m_{k+1}$  is defined by  $m_{k+1}(u_i) = y_i$ ,  $m_{k+1}(v_j) = w_j$  and  $du_i = 0$ ,  $dv_j = z_j$ .

*Remark 1.* In general, for a simply connected topological space  $X$  we have that  $\mathcal{A} = \mathcal{A}_{PL}(X)$  and, obviously, by (2.2), we see that  $\text{rk } \pi_{k+1}(X)$  is the number of generators in the above procedure we add to  $\mu_k(X)$ , in order to obtain  $\mu_{k+1}(X)$ .

*Remark 2.* For some spaces with special cohomology one can easily compute their minimal models. Namely, assume that the rational cohomology algebra for  $X$  is given by

$$H^*(X, \mathbb{Q}) \cong \mathbb{Q}[x_1, \dots, x_n] / \langle P_1, \dots, P_k \rangle,$$

where the polynomials  $P_1, \dots, P_k$  are without relations in  $\mathbb{Q}[x_1, \dots, x_n]$  meaning that  $\langle P_1, \dots, P_k \rangle$  is a Borel ideal. Then in [2] it is proved that such a space  $X$  is formal and its minimal model is given by

$$\begin{aligned} \mu(X) &= \mathbb{Q}[x_1, \dots, x_n] \otimes \wedge(y_1, \dots, y_k), \\ dx_i &= 0, \quad dy_i = P_i. \end{aligned}$$

Note that for a formal  $X$ , the algebra  $\mu(X) \otimes_{\mathbb{Q}} k$  coincides with the minimal model of the cohomology algebra  $(H^*(X, k), d = 0)$  for any field  $k$  of characteristic zero. The converse is also true. If there exists a field  $k$  of characteristic zero for which  $\mu(X) \otimes_{\mathbb{Q}} k$  is the minimal model for the cohomology algebra  $(H^*(X, k), d = 0)$ , then  $X$  is formal. By the result of [12], all simply connected manifolds of dimension  $\leq 7$  are formal in the sense of rational homotopy theory.

*Remark 3.* Obviously, (2.2) implies that for the purpose of calculating the ranks of the homotopy groups of  $X$  we can use  $\mu(X) \otimes_{\mathbb{Q}} \mathbb{R}$  as well. In the case of formal  $X$  it means that we can apply the above procedure to  $H^*(X, \mathbb{R})$ .

**2.1.2. Rationally elliptic spaces.** Suppose that  $X$  is a simply connected topological space with rational homology of finite type. It is said that  $X$  is *rationally elliptic* if  $\dim \pi_*(X) \otimes \mathbb{Q}$  is finite. The ranks

of the homotopy groups of a rationally elliptic space  $X$  of dimension  $n$  satisfy [6]:

$$\sum_k 2k \operatorname{rk} \pi_{2k}(X) \leq n, \quad \sum_k (2k+1) \operatorname{rk} \pi_{2k+1}(X) \leq 2n-1. \quad (2.5)$$

*Example 1.* For the spaces whose cohomology is given by Remark 2 we see that (2.2) implies that they are rationally elliptic. Among the examples of such spaces are compact homogeneous spaces of positive Euler characteristic and biquotients of compact Lie groups.

## 2.2. Geometric formality

The notion of geometric formality is introduced in [8]. A smooth manifold  $M$  is said to be *geometrically formal* if it admits Riemannian metric for which the wedge product of any two harmonic forms is again harmonic form. Recall [3] that a differential form  $\omega \in \Omega_{DR}(M)$  is harmonic if  $\Delta\omega = 0$ , where  $\Delta$  is the Laplace-de Rham operator on  $\Omega_{DR}(M)$ . Using Hodge theory it is proved [3] that any harmonic form is closed and that no harmonic form is exact. Moreover any real cohomology class for  $M$  contains unique, up to constant, harmonic representative. It implies that manifolds which are rational homology spheres are trivially geometrically formal, they have just one, up to constant, harmonic form. The Hodge theory also implies that a geometrically formal manifold is formal in the sense of rational homotopy theory. The vice versa is not true, it turns out that the notion of geometric formality is much more restrictive than the rational formality notion.

The first non-trivial and up to now the widest class of examples of geometrically formal manifolds are compact symmetric spaces [4]. They are as well one of the first examples of rationally formal spaces and the proof of their formality is based on the fact that on symmetric spaces harmonic forms related to an invariant metric form an algebra.

The non-involutive symmetries of higher order do not have any more such properties. It is proved in [10] that the generalized symmetric spaces are formal, while for most of them it is proved that they

are not geometrically formal. The examples of non-symmetric geometrically formal homogeneous spaces that are not cohomology sphere are provided in [11].

### 3. RATIONALLY ELLIPTIC MANIFOLDS OF DIMENSION FIVE OR SIX

We consider rationally elliptic manifolds of dimension 5 and 6. In dimension 5 we prove the following.

**Proposition 1.** *Let  $M$  be a smooth, compact, simply connected five-dimensional rationally elliptic manifold. Then  $M$  is rational cohomology sphere  $S^5$  or it is rational cohomology product of spheres  $S^2 \times S^3$ .*

*Proof.* Since  $M$  is simply connected and rationally elliptic for its Betti numbers it holds that  $b_4(M) = b_1(M) = 0$  and  $2b_2(M) + 4 \operatorname{rk} \pi_4(M) \leq 5$  and  $3 \operatorname{rk} \pi_3(M) + 5 \operatorname{rk} \pi_5(M) \leq 9$ . Therefore we must have

$$b_2 \leq 2 \text{ and } \operatorname{rk} \pi_3(M) \leq 3. \tag{2.1}$$

Now using described procedure for minimal model construction we obtain that  $\mu_2 = H^2(M)$  and  $m_2^{(3)} : H^3(\mu_2, \mathbb{R}) \rightarrow H^3(M, \mathbb{R})$ ,  $m_2^{(4)} : H^4(\mu_2, \mathbb{R}) \rightarrow H^4(M, \mathbb{R})$ . Since  $H^3(\mu_2, \mathbb{R}) = H^4(M, \mathbb{R}) = 0$  it follows that  $\operatorname{Im} m_2^{(3)} = 0$  and  $\operatorname{Ker} m_2^{(4)} = \mu_2^4$ . It implies that  $\mu_3 = \mu_2 \otimes \mathcal{L}(u_i, v_j)$ , where  $u_i$  correspond to the basis  $y_i$  in  $H^3(M, \mathbb{R})$ , while  $v_j$  correspond to the basis  $z_j$  in  $\mu_2^4$ . The homomorphism  $m_3$  is an extension of  $m_2$  defined by  $m_3(u_i) = y_i$  and  $m_3(v_j) = 0$ . It in particular gives that

$$\operatorname{rk} \pi_3(M) = b_3(M) + \dim \mu_2^4 = b_2(M) + \frac{b_2(b_2 + 1)}{2}. \tag{2.2}$$

Together with (2.1) it implies  $b_2(M) = 0$  or  $b_2(M) = 1$ .

For  $b_2(M) = 0$  it immediately follows that  $M$  is rational cohomology sphere  $S^5$ .

For  $b_2(M) = 1$ , using Poincaré duality we deduce that  $M$  is rational cohomology product of spheres  $S^2 \times S^3$ .

Any six-dimensional simply-connected rationally elliptic space  $M$  satisfies  $b_1(M) = b_5(M) = 0$  and

$$2b_2(M) + 4 \operatorname{rk} \pi_4(M) + 6 \operatorname{rk} \pi_6(M) \leq 6, \quad 3 \operatorname{rk} \pi_3(M) + 5 \operatorname{rk} \pi_5(M) \leq 11.$$

This implies that

$$b_2 \leq 3, \quad \operatorname{rk} \pi_4(M) \leq 1, \quad \operatorname{rk} \pi_3(M) \leq 3. \quad (2.3)$$

When the second Betti number is one or zero we prove:

**Proposition 2.** *Let  $M$  be a smooth, compact, simply-connected and rationally elliptic six-manifold whose second Betti number is less or equal one. Then  $M$  is rational cohomology sphere  $S^6$ , the product of spheres  $S^2 \times S^4$  or  $S^3 \times S^3$ , the complex projective space  $\mathbb{C}P^3$ .*

*Proof.* If  $b_2 = 0$  then  $b_4 = 0$  and Hurewicz theorem gives that  $\operatorname{rk} \pi_3 = b_3$ . Since, by Poincaré duality,  $b_3(M)$  has to be even we might have  $b_3 = 0$  or  $b_3 = 2$ . For  $b_3(M) = 0$  all Betti numbers  $b_i(M) = 0$ ,  $1 \leq i \leq 4$  are trivial what implies that  $M$  is rational cohomology sphere  $S^6$ . For  $b_3 = 2$  the Poincaré duality implies that  $M$  is rational cohomology product of spheres  $S^3 \times S^3$ .

For  $b_2 = 1$  we claim that  $b_3 = 0$ . First  $\mu_3 = \mathfrak{L}(x) \otimes \mathfrak{L}(u_i, v)$ , where  $x$  is the generator for  $H^2(M, \mathbb{R})$ , then  $u_i$  correspond to the basis for  $H^3(M)$  and  $v$  corresponds to the generator  $z$  for  $\operatorname{Ker} m_2^4$ . Also the differential  $d$  in  $\mu_3$  is given by  $d(u_i) = 0$  and  $d(v) = z$ . It means that  $\operatorname{rk} \pi_3(M) \geq b_3(M)$  implying  $b_3(M) \leq 2$ . For  $b_3(M) = 2$ , since  $b_5(M) = 0$ , we would have that  $\operatorname{Ker} m_3^{(5)} = H^5(\mu_3, d) = \mathfrak{L}(xu_1, xu_2)$  is two-dimensional what would imply that  $\operatorname{rk} \pi_4(M) \geq 2$ . This contradicts with (2.3).

Thus, let  $b_2 = 1$  and  $x \in H^2(M, \mathbb{R})$ . If  $x^2 = 0$ , we have in  $H^*(M)$  a generator of degree 4 and thus in this case  $M$  is rational cohomology  $S^2 \times S^4$ . If  $x^2 \neq 0$ , then  $M$  is rational cohomology complex projective space  $\mathbb{C}P^3$ .



#### 4. GEOMETRIC FORMALITY AND RATIONAL ELLIPTICITY IN DIMENSION SIX

We consider in this section geometrically formal six-dimensional rationally elliptic manifolds whose second Betti number is 2 or 3. For those whose second Betti number is 2 we obtain rational cohomology description, while among those whose second Betti number is 3, we study the class of homogeneous spaces and biquotients.

**Theorem 1.** *Let  $M$  be a smooth, compact, simply connected rationally elliptic six-manifold such whose second Betti number is 2. If  $M$  is geometrically formal then  $M$  is rational cohomology  $S^2 \times \mathbb{C}P^2$ .*

*Proof.* Let  $x$  and  $y$  be the generators in  $H^*(M)$  of degree 2. By (2.3) we have that  $\text{rk } \pi_4(M) = 0$  what implies that there is no generator of degree four in the cohomology ring  $H^*(M, \mathbb{R})$ . Since  $b_2(M) = b_4(M)$  it further gives that there is exactly one relation in  $H^*(M)$  of degree 4. Therefore  $\mu_3 = H^2(M, \mathbb{R}) \otimes \mathfrak{L}(u_i, v)$ , where  $u_i$  correspond to the basis  $y_i$  for  $H^3(M)$  while  $v$  correspond to the non-zero element  $z$  from  $\text{Ker } m_2^{(4)}$ . We also have that  $m_3(u_i) = y_i$ ,  $m_3(v) = 0$  and the differential  $d$  in  $\mu_3$  is given by  $d(u_i) = 0$ ,  $d(v) = z$ . Thus  $\text{Im } m_3^{(4)} = H^4(M, \mathbb{R})$  and  $\text{Ker } m_3^{(5)} = \mathfrak{L}(xu_i, yu_i)$ , where  $x, y$  are the generators for  $H^2(M, \mathbb{R})$  which means that the dimension of  $\text{Ker } m_3^{(5)}$  is  $2b_3(M)$ . It gives that  $\mu_4 = \mu_3 \otimes \mathfrak{L}(w_i)$ , where  $w_i$  correspond to the generators for  $\text{Ker } m_3^{(5)}$  and consequently  $\text{rk } \pi_4(M) = 2b_3(M)$ . Therefore we obtain  $b_3(M) = 0$ . The elements  $x^3, x^2y, xy^2, y^3$  are in  $H^6(M, \mathbb{R})$  which is one-dimensional. Two relations among these elements come from the relation in  $H^4(M, \mathbb{R})$  and, thus, we must have exactly one new relation in  $H^6(M, \mathbb{R})$ .

The following cases are possible.

a) There exist generator of degree 2 whose square is zero. In this case Poincaré duality implies that  $M$  has cohomology of  $S^2 \times \mathbb{C}P^2$ .

b) There is no generator of degree 2 whose square is zero. In this case we prove that one can always find generators  $\bar{x}$  and  $\bar{y}$  for  $H^2(M)$  such that

$$\bar{x}^2 + \epsilon \bar{y}^2 = 0 \text{ and } \bar{y}^3 = 0, \text{ for } \epsilon = \pm 1. \tag{2.1}$$

It would then imply that  $\bar{x}^2y = 0$  and  $\bar{x}^3 \neq 0$ .

In the cohomology ring  $H^*(M, \mathbb{R})$  there are exactly two independent relations, one in degree four and the other one in degree six. The relation in degree four is of the form  $ax^2 + bxy + cy^2 = 0$ , where  $a^2 + b^2 + c^2 \neq 0$ . We differentiate the two cases.

1) If  $a = c = 0$  this relation writes as  $xy = 0$ . It implies that  $x^2y = xy^2 = 0$  and  $x^2, y^2$  are going to be Poincaré duals to  $x$  and  $y$  respectively what gives  $x^3, y^3 \neq 0$ . Thus, the relation of degree six is of the form  $y^3 = ax^3, a \neq 0$ . Put  $x_1 = \sqrt[3]{a}x$  and consider the new generators for  $H^*(M, \mathbb{R})$  given by  $\bar{x} = x_1 + y$  and  $\bar{y} = x_1 - y$ . Then  $\bar{x}^2 - \bar{y}^2 = 0$  and  $\bar{y}^3 = x_1^3 - y^3 = 0$ , satisfying (2.1).

2) If  $a^2 + c^2 \neq 0$  let us, without loss of generality assume that  $a \neq 0$ . The relation of degree four writes as  $x^2 + bxy + cy^2 = 0$  for some new  $b$  and  $c$  what gives  $(x + \frac{b}{2}y)^2 + (c - \frac{b^2}{4})y^2 = 0$ . Note that  $c - \frac{b^2}{4} \neq 0$  since otherwise we would have the generator  $x_1 = x + \frac{b}{2}y$  whose square is zero. Thus,  $x_1$  and  $y$  satisfy in degree four the relation  $x_1^2 + by^2 = 0$  for some new  $b \neq 0$ . If we further take  $y_1 = \sqrt{|b|}y$  we obtain the generators  $x_1$  and  $y_1$  for  $H^*(M, \mathbb{R})$  related in degree four by  $x_1^2 \pm y_1^2 = 0$ . Without loss of generality assume that  $x_1^2 + y_1^2 = 0$ . It implies  $x_1^3 = -x_1y_1^2$  and  $y_1^3 = -x_1^2y_1$ . If  $x_1^3 = 0$  or  $y_1^3 = 0$  we take  $\bar{x}$  and  $\bar{y}$  to be  $x_1$  and  $x_2$ , and (2.1) will be satisfied. If  $x_1^3, y_1^3 \neq 0$  we will show that for some  $a \in \mathbb{R}$  the cube of element  $x + ay$  has to be zero. Namely, taking into account relations between  $x$  and  $y$  we obtain

$$(x + ay)^3 = (1 - 3a^2)x^3 + (a^3 - 3a)y^3.$$

Further in this case the relation in degree six writes as  $y^3 = \alpha x^3$  what gives

$$(x + ay)^3 = (1 - 3a^2 + \alpha(a^3 - 3a))x^3.$$

Therefore  $(x + ay)^3 = 0$  if and only if  $1 - 3a^2 + \alpha(a^3 - 3a) = 0$ . The later one equation, being cube equation in  $a$ , always has at least one solution. Take  $\bar{x} = ax - y$  and  $\bar{y} = x + ay$ . Then

$$\bar{x}^2 + \bar{y}^2 = (a^2 + 1)(x^2 + y^2) = 0, \quad \bar{y}^3 = 0.$$

We continue the proof. If we assume  $M$  to be geometrically formal, the relations (2.1) between the cohomology classes will be satisfied for the harmonic forms  $\omega_1$  and  $\omega_2$  representing  $x$  and  $y$ . It would imply that the form  $\omega_2$  has non trivial kernel meaning that there locally exists non-trivial vector field  $v$  such that  $i_v\omega_2 = 0$ . It would also follow that  $\omega_1^3$  is a volume form on  $M$ . But then from (2.1) we deduce that  $i_v(\omega_1^2) = 0$  as well implying that  $i_v(\omega_1)\omega_1 = 0$ . Therefore

$$i_v(\omega_1^3) = 3i_v(\omega_1)\omega_1^2 = 0$$

what is in the contradiction with the fact that  $\omega_1$  is a volume form on  $M$ . Thus such  $M$  can not be geometrically formal.

*Remark 4.* Note that a manifold  $M$  which satisfies conditions of Theorem 1 and which is not rational cohomology  $S^2 \times \mathbb{C}P^2$  can not be geometrically formal for cohomological reasons. It implies that such  $M$  may not have the cohomology of a symmetric space.

#### 4.1. Homogeneous spaces and biquotients.

It is known [6] that homogeneous spaces and biquotients of a compact Lie group are rationally elliptic. Together with Theorem 1 it implies:

**Corollary 1.** *Simply connected six-dimensional homogeneous space or biquotient whose second Betti number is two and which is not cohomology  $S^2 \times \mathbb{C}P^2$  can not geometrically formal.*

Examples of spaces which satisfy the conditions in Corollary 1 are, among the others, flag manifold  $SU(3)/T^2$  and Eschenburg's biquotients [5]. Therefore these spaces are not geometrically formal. We want to note that for the flag manifold and some of Eschenburg's biquotients it is proved in [10] and [11] that they are not geometrically formal. It is done treating separately each of these examples by studying their cohomology structure. By Corollary 1 we provide general proof for all of them.

**Lemma 1.** *Let  $M$  be a simply connected rationally elliptic six-dimensional manifold. If  $b_2(M) = 3$  then  $M$  has the rational homotopy groups of  $S^2 \times S^2 \times S^2$ .*

*Proof.* The assumption that  $b_2(M) = 3$  and that  $M$  is rationally elliptic implies that  $\text{rk } \pi_2(M) = 3$ , and  $\text{rk } \pi_{2k}(M) = 0$  for  $k \geq 2$ . We further obtain that  $\text{Im } m_2^{(3)} = 0$  and since  $b_4(M) = 3$  we must have  $\dim \text{Ker } m_2^{(4)} \geq 3$ , what implies that  $\text{rk } \pi_3 \geq 3$ . It then follows from (2.2) that  $\text{rk } \pi_3(M) = 3$  and  $\text{rk } \pi_{2k+1}(M) = 0$  for  $k \geq 2$ . Thus  $M$  has the rational homotopy groups of  $S^2 \times S^2 \times S^2$ .

By the recent results obtained in [13] and [7] it follows that a biquotient  $M$  which has the same rational homotopy groups as  $S^2 \times S^2 \times S^2$  is diffeomorphic to the biquotient of  $SU(2) \times SU(2) \times SU(2)$  by some free linear action of  $S^1 \times S^1 \times S^1$ . On the other hand the result of [7] says that any free linear action of  $T^3$  on  $(S^3)^3$ , up to reparametrization, is given by

$$(u, v, w) * ((p_1, p_2), (q_1, q_2), (r_1, r_2)) = \\ ((up_1, u^{a_1}v^{a_2}w^{a_3}p_2), (vq_1, u^{b_1}v^{b_2}w^{b_3}q_2), (wr_1, u^{c_1}v^{c_2}w^{c_3}r_2)),$$

where  $a_1, b_2, c_3 = \pm 1$  and the  $2 \times 2$  minors around the diagonal of the following matrix

$$\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$$

as well as the matrix itself have determinant  $\pm 1$ . It was shown in [13] that the conditions on the entries of this matrix are necessary and sufficient conditions for this action to be free. In [7] all such matrices are classified and there are obtained, up to equivalences, three infinite families of matrices and 12 sporadic examples. The infinite families of matrices are:

$$\begin{bmatrix} 1 & 2 & 0 \\ 1 & 1 & 0 \\ c_1 & c_2 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 2 & a_3 \\ 1 & 1 & b_3 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ b_1 & 1 & 0 \\ c_1 & c_2 & 1 \end{bmatrix}.$$

The action of  $T^3$  on  $(S^3)^3$  given by the second matrix produces Tataro's biquotients studied in [13] as an example of family of 6-manifolds with nonnegative sectional curvature, but with infinitely many distinct classes of rational cohomology rings. For them, it is proved in [11] not to be geometrically formal for cohomological reasons.

We prove here that the family of biquotients given by the third matrix is not geometrically formal for cohomological reasons.

**Theorem 2.** *The biquotients  $(S^3)^3/T^3$  given by the action*

$$(u, v, w) * ((p_1, p_2), (q_1, q_2), (r_1, r_2)) = ((up_1, up_2), (vq_1, u^{b_1}vq_2), (wr_1, u^{c_1}v^{c_2}wr_2)),$$

are not geometrically formal.

*Proof.* Using the standard techniques [13] for computing cohomology rings one deduce [7] that any biquotient  $M$  obtained by the given action has the following integral cohomology structure :

$$H^*(M) = \mathbb{Z}[x_1, x_2, x_3] / \langle x_1^2 = 0, x_2^2 = -b_1x_1x_2, x_3^2 = -c_1x_1x_3 - c_2x_2x_3 \rangle$$

Let us assume that  $M$  admits a formal metric. Denote by  $\omega_1, \omega_2$  and  $\omega_3$  the corresponding harmonic representatives of the real cohomology generators  $x_1, x_2$  and  $x_3$ . Then these harmonic forms will satisfy the relations which hold in  $H^*(M, \mathbb{R})$  between  $x_1, x_1$  and  $x_3$ :

$$\omega_1^2 = 0, \quad \omega_2^2 = -b_1\omega_1\omega_2, \quad \omega_3^2 = -c_1\omega_1\omega_3 - c_2\omega_2\omega_3. \tag{2.2}$$

It implies that  $(\frac{b_1}{2}\omega_1 + \omega_2)^2 = 0$  and we further consider the harmonic form  $\tilde{\omega}_2 = \omega_2 + \frac{b_1}{2}\omega_1$  which satisfies  $\tilde{\omega}_2^2 = 0$ . Thus  $\omega_3^2 = (\frac{b_1}{2}c_2 - c_1)\omega_1\omega_3 - c_2\tilde{\omega}_2\omega_3$  and we consider the form

$$\tilde{\omega}_3 = \omega_3 - \frac{1}{2}(\frac{b_1}{2}c_2 - c_1)\omega_1 + \frac{1}{2}\tilde{\omega}_2,$$

whose square satisfies

$$\tilde{\omega}_3^2 = -\frac{c_2}{2}(\frac{b_1}{2}c_2 - c_1)\omega_1\tilde{\omega}_2. \tag{2.3}$$

Therefore the form  $\omega_1\tilde{\omega}_2\tilde{\omega}_3$  is a volume form on  $M$  and such is  $\tilde{\omega}_3^3 = -\frac{c_2}{2}(\frac{b_1}{2}c_2 - c_1)\omega_1\tilde{\omega}_2\tilde{\omega}_3$  as well.

The kernel foliations for the forms  $\omega_1$  and  $\tilde{\omega}_2$  are each of dimension at least four, so they have a common vector field  $v$  (more precisely at least two) meaning that  $i_v(\omega_1) = i_v(\tilde{\omega}_2) = 0$ . Together with (2.3) it implies

$$i_v(\tilde{\omega}_3^2) = 2i_v(\tilde{\omega}_3)\tilde{\omega}_3 = 0$$

and, thus,

$$i_v(\tilde{\omega}_3^3) = 3i_v(\tilde{\omega}_3)\tilde{\omega}_3^2 = 0,$$

what is in contradiction with the fact that  $\tilde{\omega}_3^3$  is a volume form on  $M$ .

## REFERENCES

1. C. Baer, *Geometrically formal 4-manifolds with nonnegative sectional curvature*, arXiv: 1212.1325, (2013).
2. A. Bousfield and V. Gugenheim, *On PL de Rham theory and rational homotopy type*, Mem. Amer. Math. Soc. **179** (1976), ix–94.
3. S. S. Chern, *Complex manifolds*, The University of Chicago, 1956.
4. B. A. Dubrovin, A. T. Fomenko and S. P. Novikov, *Modern Geometry — Methods and Applications*, Vol. III, Graduate Texts in Mathematics 124, Springer Verlag 1990.
5. J. H. Eschenburg, *Cohomology of biquotients*, Manuscripta Math. **75** (1992), 151–166.
6. Y. Felix, S. Halperin and J. C. Thomas, *Rational Homotopy Theory*, Springer Verlag, 2000.
7. J. DeVito, *The classification of simply connected biquotients of dimension at most 7 and 3 new examples of almost positively curved manifolds*, Ph.D.-thesis, University of Pensilvania, ProQuest LLL, 2011.
8. D. Kotschick, *On products of harmonic forms*, Duke Mathematical Journal **107** (2001), no. 3, 521–532.
9. D. Kotschick, *Geometric formality and non-negative scalar curvature*, arXiv:1212.3317, (2013).

10. D. Kotschick and S. Terzić, *On formality of generalised symmetric spaces*, Mathematical Proceedings of the Cambridge Philosophical Society **134** (2003), 491–505.
11. D. Kotschick and S. Terzić, *On geometric formality of homogeneous spaces and of biquotients*, Pacific Journal of Mathematics **249** (2011), no. 1, 157–176.
12. T. Miller, J. Neisendorfer, *Formal and coformal spaces*, Illinois Journal of Mathematics, **22** (1978), 565–580.
13. B. Totaro, *Curvature, diameter, and quotient manifolds*, Mathematical Research Letters **10** (2003), 191–203.