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*Žana Kovijanić Vukićević**

KÖNIG'S GRAPHS IN THE PROBLEM OF LINEAR INDEPENDENCE OF MATRICES

Abstract

This paper proves conjecture (see [1]) of linear independence for special type of $(0,1)$-matrices that appear in Brauer's representation of Temperley-Lieb Algebras, considering, instead of matrices, the appropriate graphs assigned to them.

KÖING-OVI GRAFOVI U PROBLEMU LINEARNE NEZAVISNOSTI MATRICA

Sažetak

U radu je dokazana pretpostavka ([1]) o linearnoj nezavisnosti jednog tipa $(0,1)$-matrica koje se pojavljuju u Brauerovoj reprezentaciji Temperley-Lieb algebri. (0,1)-matricama su pridruženi odgovarajući bipartitni grafovi, pa je pitanje linearne nezavisnosti matrica nadalje tretirano aparatom teorije grafova.

* Department of Mathematics, University of Montenegro, Podgorica, Montenegro

1. INTRODUCTION

Let A_2 be the matrix

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

and A_2^T its transpose. The matrix $a_2 = A_2 A_2^T$ is

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}.$$

For each $n \in N$ and each $k \in \{0, 1, \dots, n-1\}$ let

$$h_k^n = \mathbf{1}_{2^{n-k-1}} \otimes a_2 \otimes \mathbf{1}_{2^{k-1}},$$

where $\mathbf{1}_n$ is $n \times n$ identity matrix and $\otimes$ is the Kronecker product of matrices. Matrices $\mathbf{1}_{2^n}, h_0^n, h_1^n, \cdots, h_{n-1}^n$ are **elementary n-matrices** . Denote by $\mathcal{M}_n$ the set of all matrices that can be obtained as a finite product of some, not necessary distinct, elementary n-matrices. Let $\sim$ be a binary relation over $\mathcal{M}_n$ such that for each two matrices $A, B \in \mathcal{M}_n$:

$A \sim B$ *iff* there is a rational number α such that $A = \alpha B$

This paper proves conjecture (see [1]) of linear independence for matrices from $\mathcal{M}_{n/\sim}$. It will be done by listing them all, and by finding for each an entry with 1 where the others in front of it have 0.

2. KÖNIG'S GRAPHS

In this section we introduce König's graphs, a well known graph-theoretic representation of matrices (see [2]). It will make it possible for us to prove the main theorem of linear independence considering, instead of matrices, the appropriate graphs assigned to them.

To each matrix $\mathbf{A}$ assign a complete bipartite weight-graph $G(\mathbf{A})$ on 2n vertices colored by black and white color. Black vertices are in one-to-one correspondence with the rows of matrix $\mathbf{A}$ and they are labelled by $i_\bullet$, $i = \overline{1,n}$. White vertices correspond to the columns of matrix $\mathbf{A}$ and they are labelled by $i_\circ$, $i = \overline{1,n}$. To each edge $\rho = (i_\bullet, j_\circ)$ assign (i,j)-entry of matrix $\mathbf{A}$. It is called a weight of edge ρ. Bipartite graph G(A) is a König's graph of matrix A.

Let G_1 and G_2 be some König's graphs. Composition of the graphs G_1 and G_2 is a 3-partite graph $G_1 * G_2$ obtained from the G_1 and G_2 identifying each white vertex $i_\circ$ of graph G_1 with the black vertex $i_\bullet$ of graph G_2. Vertices obtained by identification are gray.

Product of the König's graphs G_1 and G_2 is a complete bipartite weight-graph G_1G_2 obtained from $G_1 * G_2$ deleting all of its edges and gray vertices and adding new edges between black and white vertices. The weight of edge $(i_\bullet, j_\circ)$ is the sum of weights of all paths length two between $i_\bullet$ and $j_\circ$ in $G_1 * G_2$, while the weight of path is the product of weights of edges belonging to that path.

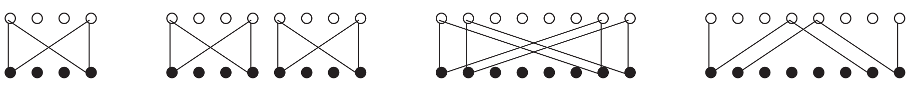
The next lemma provides the question whether the set $\mathcal{M}_{n/\sim}$ is linearly independent or not, to be considered using the König's graphs of elementary matrices and their products.

Theorem 1 *([2]) Let A and B be square matrices of the same type. Then*

$$G(AB) = G(A)G(B).$$

In the sequel, matrices and their König's graphs will be identified and denoted by the same symbols. The useful property of König's graphs is that the edges with weight 0 may be removed. Further, if the all nonzero entries of matrix are 1, as in the our case, the weights will be no emphasized in the figure of the graph.

König's graphs of matrices a_2, h_1^3, h_2^3 and $h_2^3h_1^3$ are illustrated in the next figure.



a_2 $\quad$ h_1^3 $\quad$ h_2^3 $\quad$ $h_2^3h_1^3$

Each square matrix may be imagined as square properly filled by some numbers. Identity matrices and matrix a_2 are centrally symmetric about the center of own imagined square and, furthermore for each matrix A holds: $\mathbf{1}_{2^2} \otimes A = A \oplus A$, where $\oplus$ is sum of matrices defined as follows

$$\left[\begin{array}{c|c} \mathrm{A} & 0 \\ \hline 0 & \mathrm{A} \end{array}\right].$$

The König's graph of the sum $A \oplus A$ is the union of two copies of graph A defined in standard way.

Hence, from the definition of Kronecker product $\otimes$ we have that each matrix A from $\mathcal{M}_n$ is centrally symmetric about the center of own imagined square. Equivalently, if $\rho = (i_\bullet, j_\circ)$ as an edge of the König's graph A and for each $i \in \{1, 2, \ldots, 2^n\}$, $\mathcal{S}_n(i) = 2^n - i + 1$ is a number symmetric to i about the number $\frac{2^n+1}{2}$, then

$$\mathcal{S}_n(\rho) = (\mathcal{S}_n(i)_\bullet, \mathcal{S}_n(j)_\circ)$$

is an edge of the same graph. This property will be called the **symmetry about the vertical axis**.

3. NORMAL FORMS IN $\mathcal{M}_n$

Definition

The essential part of the proof of linear independence in $\mathcal{M}_{n/\sim}$ is the definition of unique normal form for its elements.

For $1 \le j \le i \le n-1$ let the **block** $\mathbf{h^n_{[i,j]}}$ be defined as

$$h^n_i h^n_{i-1} \ldots h^n_{j+1} h^n_j$$

We say that an product from $\mathcal{M}_n$ is in **normal form** if it is either the identity matrix or it looks as follows:

$$h^n_{[b_1,c_1]} h^n_{[b_2,c_2]} \ldots h^n_{[b_m,c_m]}$$

where $b_1 < b_2 < \ldots < b_m$ and $c_1 < c_2 < \ldots < c_m$.

It is known ([1]) that:

$$h_i h_j = h_j h_i, \quad \text{for } | i - j | \geq 2, \tag{3.1}$$

$$h_i h_{\pm 1} h_i = h_i \tag{3.2}$$

$$h_i h_i = 2h_i \tag{3.3}$$

Then, by [1], we have the next lemma:

Lemma 1 *Every element of $\mathcal{M}_n$, i.e. every finite product of elementary n-matrices, is equivalent with a product in normal form.*

Set of Kónig's graphs of normal forms will be denoted by $\mathcal{T}_n$. We are going to list them all and to find for each an edge that the others in front it in the list have no.

Linear order over the set $\mathcal{T}_n$.

We first introduce some terminology and notations for later use.

For each $n \in N, n \geq 2$, let $L^n = \{1, 2, \ldots, 2^{n-1}\}$ and $R^n = \{2^{n-1} + 1, 2^{n-1} + 2, \ldots, 2^n\}$. L^n is a left half of the set $\{1, 2, \ldots, 2^n\}$ and R^n is the right half of that set.

The edge $\rho = (i_\bullet, j_\circ)$ is the **long edge** in some graph if its endpoints belong to distinct halves of the set $\{1, 2, \ldots, 2^n\}$.

Denote by S_0^n the left sight of L^n, that is the set $\{1, 2, \ldots, 2^{n-2}\}$, and by X_0^n the right half of L^n, that is the set $\{2^{n-2} + 1, 2^{n-2} + 2, \ldots, 2^{n-1}\}$. For each $1 \leq k \leq n - 2$ let:

$$X_k^n = \begin{cases} \text{left half of } X_{k-1}^n & , \quad \text{k is odd} \\ \text{right half of } X_{k-1}^n & , \quad \text{k is even} \end{cases}$$

and

$$S_k^n = \begin{cases} \text{right half of } X_{k-1}^n & , \quad \text{k is odd} \\ l \text{ left half of } X_{k-1}^n & , \quad \text{k is even.} \end{cases}$$

We can check easily that for any k, $1 \leq k \leq n - 2$, worth:

$$| S_k^n |=| X_k^n |= 2^{n-k-2}, \tag{3.4}$$

$$X_1^n \supset X_2^n \supset \ldots \supset X_{n-2}^n, \tag{3.5}$$

$$S_k^n \cap X_k^n = \emptyset, \tag{3.6}$$

$$S_k^n \subset X_{k-1}^n = \emptyset, \tag{3.7}$$

For each $0 \leq k \leq n-2$, by C_k^n and Y_k^n will be denoted subsets of the set $\{1, 2, \ldots, 2^n\}$ centrally symmetric to S_k^n and X_k^n about the number $\frac{2^n+1}{2}$, that is

$$C_k^n = \{\mathcal{S}_n(i) \mid i \in S_k^n\} \quad \text{and} \quad Y_k^n = \{\mathcal{S}_n(i) \mid i \in X_k^n\}.$$

Then, C_0^n is the right half of R^n, that is the set $\{3 \cdot 2^{n-2}+1, 3 \cdot 2^{n-2}+2, \ldots, 2^n-\}$, Y_0^n the left half of R^n, that is the set $\{2^{n-1}+1, 2^{n-1}+2, \ldots, 3 \cdot 2^{n-2}\}$ and for each $1 \leq k \leq n-2$:

$$Y_k^n = \begin{cases} \text{right half of } Y_{k-1}^n & , \quad \text{k is odd} \\ \text{left half of } Y_{k-1}^n & , \quad \text{k is even} \end{cases}$$

$$C_k^n = \begin{cases} \text{left half of } Y_{k-1}^n & , \quad \text{k is odd} \\ \text{right half of } Y_{k-1}^n & , \quad \text{k is even.} \end{cases}$$

Then it is easy establish the following: for each $n \in N$, $R^n = X_0^{n+1}$ and for each $k \leq n-2$

$$C_k^n = S_{k+1}^{n+1} \tag{3.8}$$

The whole construction has made so that the next is satisfied: for any $n-$elementary graph h_i^n there is no edge with white vertex belonging to the union $X_{n-i-1}^n \cup Y_{n-i-1}^n$.

Next, based on the Normal Form Lemma, we make a sequence of families $\mathcal{H}^n \subseteq \mathcal{T}_n$, $n \geq 2$ and define linear order $<$ over $\mathcal{H}^n$ as follows.

Let $\mathcal{H}^2 = \{\mathbf{1}_{2^2}, h_1^2\}$ and $\mathbf{1}_{2^2} < h_1^2$.

For $n \geq 2$ assume that $\mathcal{H}^n \subseteq \mathcal{T}_n$ is linearly ordered by relation $<$. We proceed to make $\mathcal{H}^{n+1} \subseteq \mathcal{T}_{n+1}$. It will be done in tree steps for $n+1=3$ and in four steps for $n+1 \geq 4$.

step one: Let

$$H_0^{n+1} = \left\{ h_{[b_1,c_1]}^{n+1} h_{[b_2,c_2]}^{n+1} \ldots h_{[b_m,c_m]}^{n+1} \mid b_m \leq n-1 \right\}$$

or equivalently,

$$H_0^{n+1} = \{f \oplus f \mid f \in \mathcal{H}^n\}.$$

Linear order over the se H_0^{n+1} is defined by taking that

$$f \oplus f < g \oplus g \text{ iff } f < g,$$

for each $f, g \in \mathcal{H}^n$.

step two: Let

$$H_1^{n+1} = \left\{ f' h_{[n,n]}^{n+1} \mid f' \in H_0^{n+1} \right\},$$

that is

$$H_1^{n+1} = \left\{ (f^* \oplus f^*) h_{[n,n]}^{n+1} \mid f^* \in \mathcal{H}^n \right\},$$

Linear order over the union $H_0^{n+1} \cup H_1^{n+1}$ is defined as extension of the order $<$ over the family H_0^{n+1} and by taking that for each $f, g \in H_1^{n+1}$

$$f < g \text{ iff } f^* < g^*$$

and for each $f \in H_0^{n+1}$ and each $g \in H_1^{n+1}$

$$f < g.$$

step three: Let

$$H_2^{n+1} = \left\{ h_{[b_1,c_1]}^{n+1} \dots h_{[b_{m-1},c_{m-1}]}^{n+1} h_{[n,c_m]}^{n+1} \mid c_m \le n-1, b_{m-1} \le n-2 \right\}$$

and for each $f \in H_0^{n+1} \cup H_1^{n+1}$ and each $g \in H_2^{n+1}$

$$f < g.$$

Further, for any

$$f = h_{[b_1,c_1]}^{n+1} \dots h_{[b_{m-1},c_{m-1}]}^{n+1} h_{[n,c_m]}^{n+1} \in H_2^{n+1},$$

let

$$f' = h_{[b_1,c_1]}^{n+1} \dots h_{[b_{m-1},c_{m-1}]}^{n+1} h_{[n-1,c_m]}^{n+1}$$

be obtained from f omitting the elementary matrix h_n^{n+1}. Then $f' = f^* \oplus f^*$, for $f^* = h^n_{[b_1,c_1]} \dots h^n_{[b_{m-1},c_{m-1}]} h^n_{[n-1,c_m]}$ Using this notation, we say that for each $f, g \in H_2^{n+1}$

$$f < g \text{ iff } f^* < g^*$$

step four: If $n+1 \geq 4$ for each $k \in N$, $2 \leq k \leq n-1$, let

$$H_{3,k}^{n+1} = \Big\{ f' h^{n+1}_{[n,k]} \mid \quad f' \in H_0^{n+1} \text{ and the last block in f' is } h^{n+1}_{[n-1,c]}, c \leq k-1 \Big\}$$

Finally, let

$$H_3^{n+1} = \bigcup_{k=2}^{n-1} H_{3,k}^{n+1}$$

and

$$\mathcal{H}^{n+1} = H_0^{n+1} \cup H_1^{n+1} \cup H_2^{n+1} \cup H_3^{n+1}.$$

Let us extent linear order $<$ defined over the union $H_0^{n+1} \cup H_1^{n+1} \cup H_2^{n+1}$ in the first tree steps onto the $\mathcal{H}^{n+1}$ as follows:

if $f \in H_0^{n+1} \cup H_1^{n+1} \cup H_2^{n+1}$ and $g \in H_3^{n+1}$ let $f < g$,

for each $i \neq j$, if $f \in H_{3,i}^{n+1}$ and $g \in H_{3,j}^{n+1}$, let $f < g$ iff $i > j$

if $f = f' h^{n+1}_{[n,i]} \in H^{n+1}_{3,i}$ and $g = g' h^{n+1}_{[n,i]} \in H^{n+1}_{3,i}$ for some $i \in \{2, 3, \dots, n-1\}$, let $f < g$ iff $f' < g'$.

By the construction and the definition of normal form, the reader will have no difficulty in showing that for each $n \in N$.

$$\mathcal{H}^n = \mathcal{T}_n.$$

4. MAIN THEOREM

The edge $\rho = (i_\bullet, j_\circ)$ of some graph $f \in \mathcal{H}^n$ will be called the **new edge** in f if ρ is no edge of any graph $g \in \mathcal{H}^n$ less than f.

Theorem 2 *Let $n \in N$, $n \geq 2$. In any König's graph $f \in \mathcal{H}^n$ there are new edges ρ and $\hat{\rho}$ such that if the last block of graph f is $h^n_{[n-1,s]}$ then ρ is the long edge and its white vertex $j_\circ$ belongs to the set S^n_{n-1-s} and $\hat{\rho}$ is the long edge and its white vertex belongs to the set C^n_{n-1-s}.*

Proof: The proof is by induction on $n \geq 2$. The basis of induction is true by definition of the family $\mathcal{H}^2$ and linear order over it. Let us assume that $n \geq 2$ and the assertion is true for any graph of family $\mathcal{H}^n$. We prove that in any $f \in \mathcal{H}^{n+1}$ there are new edges $\rho = (i_\bullet, j_\circ)$ and $\hat{\rho} = (l_\bullet, m_\circ)$ such that if the last block of graph f is $h^{n+1}_{[n,s]}$ then ρ is the long edge and its white vertex $j_\circ$ belongs to the set S^{n+1}_{n-s} and $\hat{\rho}$ is the long edge and its white vertex $m_\circ$ belongs to the set C^{n+1}_{n-s}. As the constructing procedure of the family $\mathcal{H}^{n+1}$, the proof is going to be developed in four steps:

step one: For $f \in H^{n+1}_0$ the claim is true by inductive hypothesis.

step two: Let $f \in H^{n+1}_1$. Then,

$$f = (f^* \oplus f^*)h^{n+1}_{[n,n]}$$

for some $f^* \in \mathcal{H}^n_0$ and there is edge $\rho^* = (i_\bullet, j_\circ)$ so that ρ^* is the new edge in f^*. Because of the **property of symmetry about the vertical axis** , we may suppose that $j_\circ \in S^n_0 = \{1, 2, \dots, 2^{n-2}\}$, or $j_\circ \in Y^n_0 = \{2^{n-1}+1, 2^{n-1}+1, \dots, 3 \cdot 2^{n-2}\}$.

In the first case, when $j_\circ \in S^n_0 = \{1, 2, \dots, 2^{n-2}\}$, let $k = 3 \cdot 2^{n-1} + j$. Then, $(j_\bullet, k_\circ)$ is an edge of $(n+1)-$elementary graph h^{n+1}_n and hence $\rho = (i_\bullet, k_\circ)$ is an edge of f.

Let us prove that ρ is new edge in f.

If $g \in H^{n+1}_0$, ρ is no edge of g because of it is a long edge and that type of edges doesn't exist in any graph of family H^{n+1}_0.

Let $g = (g^* \oplus g^*)h^{n+1}_{[n,n]}$ and $g < f$. By the definition of products of König's graphs and recalling what are the edges of elementary graph h^{n+1}_n, $\rho = (i_\bullet, k_\circ)$ is an edge of graph g if and only if either $(i_\bullet, k_\circ)$ or $(i_\bullet, j_\circ)$ is in the graph $g^* \oplus g^*$. The first is no true because of it is a long edge. The second is impossible because of the choice of edge $\rho^* = (i_\bullet, j_\circ)$ as a new edge in f^*.

Therefore, $\rho = (i_\bullet, k_\circ)$ is a new long edge whose white vertex $k_\circ$ belongs to the set C_0^{n+1}. Then, $\mathcal{S}_{n+1}(\rho)$ is a new long edge whose white vertex belongs to the set S_0^{n+1}.

In the second case, when $j_\circ \in Y_0^n$, let $j' = j + 2^n$ and $i' = i + 2^n$. Then, $\rho' = (i'_\bullet, j'_\circ)$ is the new edge in $f^* \oplus f^*$. (It is the edge in the second copy of graph $f^* \in \mathcal{H}^n$ in the sum $f^* \oplus f^*$). Now, let $k = j' - 3 \cdot 2^{n-1}$. Then, $(j'_\bullet, k_\circ)$ is an edge of elementary graph h_n^{n+1} and hence $\rho = (i'_\bullet, k_\circ)$ is an edge of f. As in the previous case, it can be prove that ρ is a long new edge with the white vertex k from the set S_0^{n+1}. $\hat{\rho} = \mathcal{S}_{n+1}(\rho)$ is a long new edge with the white vertex from the set C_0^{n+1}.

step three: Let $f \in H_2^{n+1}$. Then f has a normal form

$$h_{[b_1,c_1]}^{n+1} \dots h_{[b_{m-1},c_{m-1}]}^{n+1} h_{[n,s]}^{n+1}$$

for some $s \leq n-1$ and $b_{m-1} \leq n-2$. By inequality (3.1), f is the product of König's graph h_n^{n+1} and the sum $f^* \oplus f^*$, where

$$f^* = h_{[b_1,c_1]}^{n} \dots h_{[b_{m-1},c_{m-1}]}^{n} h_{[n-1,s]}^{n}.$$

Let $\rho^* = (i_\bullet, j_\circ)$ be a new long edge in graph $f^* \in \mathcal{H}^n$ such that $i_\bullet \in L^n$ and $j_\circ \in C_{n-1-s}^n$. It exists by inductive hypothesis. Then, for $k = 3 \cdot 2^{n-1} + i$, $\rho = (k_\bullet, j_\circ)$ is an edge of the graph $f = h_n^{n+1}(f^* \oplus f^*)$. ρ is a long edge and its white vertex $j_\circ$ belongs to the set C_{n-1-s}^n that is, using inequalitu (3.8) the same set as S_{n-s}^{n+1}.

It remains to prove that ρ is a new in f. Obviously, ρ is no edge of any graph element of H_0^{n+1}. Because of the fact that $j_\circ$ belongs to the set C_{n-1-s}^n subset of R^n, ρ is no edge of any graph element of H_1^{n+1}.

To show that ρ is no edge of any graph $g \in H_2^{n+1}$, assume contrary. Then, since g is product of graph h_n^{n+1} and the sum $g^* \oplus g^*$, either $(k_\bullet, j_\circ)$ or $(i_\bullet, j_\circ)$ are edges in g^*. However, the both cases are impossible. The proof of this assertion is the same as in the previous step, so we omit it.

step four: We can prove the following lemma.

Auxiliary Lemma. For any $n \geq 2$ and $k \in \{0, 1, \ldots, n-2\}$ there is an increasing function $\alpha_k : X_k^n \to C_{k+1}^{n+1}$ such that $(i_\bullet, \alpha_k(i)_\circ)$ is edge in the graph $h_{[n,k+1]}^{n+1}$.

Proof: For any fixed $n \geq 2$ we proceed by induction on k. The basis of induction is $k = 0$. Recall that

$$X_0^n = \left\{1, 2, \ldots, 2^{n-2}\right\}.$$

For each $i \in X_0^n$ let $m_i = 3 \cdot 2^{n-1} + i$ and $\alpha_0(i) = m_i - 3 \cdot 2^{n-2}$. Then

$$\alpha_0(i) = 3 \cdot 2^{n-2} + i$$

is an element of the set

$$C_1^{n+1} = \left\{2^{n-1} + i \mid i = 1, 2, \ldots, 2^{n-2}\right\}$$

and $(i_\bullet, m_{i\circ})$ and $(m_{i\bullet}, \alpha_0(i)_\circ)$ are edges of the graphs h_n^{n+1} and h_{n-1}^{n+1}, respectively. So, $(i_\bullet, \alpha_0(i)_\circ)$ is an edge of $h_{[n,n-1]}^{n+1}$ and the function α_0 is an increasing function from X_0^n to C_1^{n+1}.

Assume that the claim is true for $k - 1 \geq 0$. We recognize the two cases: the odd and the even k.

If k is odd number, X_k^n is left half of the set X_{k-1}^n and hence $\{\alpha_{k-1}(i) \mid i \in X_k^n\}$ is left half of the set C_k^{n+1} satisfying condition

$$\mid C_k^{n+1} \mid = \mid Y_k^{n+1} \mid = 2^{n-k-1}.$$

For each $i \in X_k^n$ let

$$\alpha_k(i) = \alpha_{k-1}(i) + 3 \cdot 2^{n-k-2}.$$

Since,C_k^{n+1} is the left and Y_k^{n+1} the right half of the set Y_{k-1}^{n+1},

$$\{\alpha_k(i) \mid i \in X_k^n\}$$

is the right half of the set Y_k^{n+1}, that is the set C_{k+1}^{n+1}.

It just remains to verify the even case of k, which is quite similar with the above.

Let k be an even number. Then, X_k^n is right half of the set X_{k-1}^n and the set $\{\alpha_{k-1}(i) \mid i \in X_k^n\}$ is right half of the set C_k^{n+1}. For each $i \in X_k^n$ let

$$\alpha_k(i) = \alpha_{k-1}(i) - 3 \cdot 2^{n-k-2}.$$

Then, $\{\alpha_k(i) \mid i \in X_k^n\}$ is the left half of the Y_k^{n+1} that is the set C_{k+1}^{n+1}.

Now, let's go back to the main theorem, step four. Assume that $2 \leq k \leq n-1$. We are going to prove that in any $f \in H_{[3,k]}^{n+1}$ there is a new long edge $\hat{\rho}$ whose white vertex belongs to the set C_{n-k}^{n+1}. Then, $\rho = \mathcal{S}_{n+1}(\hat{\rho})$ will be a new long edge whose white vertex belongs to the set S_{n-k}^{n+1}.

Let $f \in H_{[3,k]}^{n+1}$. Then $f = (f^* \oplus f^*)h_{[n,k]}^{n+1}$ for some

$$f^* = h_{[b_1,c_1]}^n h_{[b_2,c_2]}^n \dots h_{[n-1,s]}^n, \quad s \leq k-1.$$

Let $\rho^* = (i_\bullet, j_\circ)$ be a new long edge in f^* satisfying condition $j_\circ \in S_{n-s-1}^n$. Because of the "length" of edge ρ^*, vertex $i_\bullet$ is an element of the set R^n. Using the inequality $S_{n-s-1}^n \subset X_{n-s-2}^n$ and inequality $X_{n-s-2}^n \subset X_{n-k-1}^n$, for each $s \leq k-1$, we obtain that $j_\circ \in X_{n-k-1}^n$. The Auxiliary Lemma guarantees that for $p = \alpha_{n-k-1}(j)$,

$$\hat{\rho} = (i_\bullet, p_\circ)$$

is an edge in graph $h_{[n,k]}^{n+1}$ such that its white vertex $p_\circ$ belongs to the set C_{n-k}^{n+1}. We will show that $\hat{\rho}$ is the edge that we are looking for.

Keeping in mind that the last block of any graph from family H_1^{n+1} is $h_{[n,n]}^{n+1}$, in any graph from that family there is no edge with white vertex from the union $X_0^{n+1} \cup Y_0^{n+1}$. Since, $p_\circ \in C_{n-k}^{n+1}$ and $C_{n-k}^{n+1} \subset Y_0^{n+1}$, $\hat{\rho}$ is no edge of that type of graphs. We proceed analogously for family H_2^{n+1}. By inequality (3.1) we have that in any graph from that family there is no edge with black vertex from the union $X_0^{n+1} \cup Y_0^{n+1}$. Since, $i_\bullet \in R^n$ and $R^n = X_0^{n+1}$, $\hat{\rho}$ is no edge of those graphs. If $k < n-1$ and $i \geq k+1$, let us consider graphs contained in $H_{3,i}^{n+1}$. The last block each of them is $h_{[n,i]}^{n+1}$, and hence, they have no edges with the white vertex belonging $X^{n+1}_{n-i} \cup Y^{n+1}_{n-i}$. Using inequality (3.5) we have that $Y^{n+1}_{n-i} \supset Y^{n+1}_{n-k-1}$ and then using inequality (3.7) $Y^{n+1}_{n-i} \supset C^{n+1}_{n-k}$. Therefore, $\hat{\rho}$ is no edge any of graph in $H^{n+1}_{3,i}$, $i \geq k+1$.

To show that $\hat{\rho} = (i_\bullet, p_\circ)$ is no edge of any graph g less then f, $g \in H^{n+1}_{3,k}$, assume contrary. Then, $g = (g^* \oplus g^*)h^{n+1}_{[n,k]}$ for some $g^* < f^*$ and either $(i_\bullet, j_\circ)$ or $(i_\bullet, (j + 3 \cdot 2^{n-1})_\circ)$ are edges in g^*. The proof that the both cases are impossible is the same as in the previous steps, so we omit it.

References

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