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STABILITY OF NON-CONSERVATIVE UNDAMPED DYNAMICAL SYSTEMS: OLD AND NEW RESULTS

Abstract

The paper deals with stability problems of linear multi-degree-of-freedom non-conservative undamped dynamical systems. Two general algebraic criteria that contain necessary and sufficient conditions for spectral stability, flutter and divergence instability are presented. Then a survey of selected simple criteria — expressed by the properties of the system's matrices — for the Lyapunov stability and instability of the systems is given. In particular, the recent generalizations of the well-known Merkin's instability theorem, as well as the results of the study of the influence of infinitesimal circulatory forces on the stability of potential systems with multiple frequencies, are reported. Several simple numerical examples are used to illustrate the usefulness of the presented results and also to compare them with each other.

Keywords: Linear system, Stability, Potential force, Circulatory force

1. INTRODUCTION

The area of stability of dynamical systems is at the crossroads of physics, mathematics and engineering. Physicists are interested in instabilities that arise in nature, the mathematicians and mechanicians are interested in exact mathematical formulations that provide conditions under which systems are stable, and the engineers are interested in the analysis and design of engineered systems so as to ensure their safe and stable behaviour.

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In recent years, there has been a resurgence of interest in the stability of linear dynamical systems, especially non-conservative ones, and new results appear. A remarkable class of such dynamical systems is associated with potential (conservative) and positional non-conservative (circulatory) forces and can be described by the equation

$$\tilde{M}\ddot{q} + \tilde{K}q + \tilde{N}q = 0 \tag{1.1}$$

where $\tilde{M}$, $\tilde{K}$ and $\tilde{N}$ are n by n constant real matrices; $\tilde{M}$ is symmetric and positive definite ($\tilde{M} = \tilde{M}^T > 0$), $\tilde{K}$ is symmetric, and $\tilde{N}$ is skew-symmetric ($\tilde{N} = -\tilde{N}^T$). $\tilde{M}$ is the mass matrix, $\tilde{K}$ describes the potential forces and $\tilde{N}$ the non-conservative forces. The n-vector of generalized coordinates is denoted by q, and the dots indicate differentiation with respect to the time, t. Derivation of Eq. (1.1) can be found in e.g. [1]. Such systems are often called non-conservative undamped or circulatory. A variety of physical and technical processes, the modelling of which results in circulatory systems, extends from self-oscillations (shimmy) in aircraft wheels, controlled motions of two-legged walking robots, and the destabilizing effect of viscous damping in bearing supports of turbine rotors to dynamics of brake squealing, flutter in aerospace systems, magneto-hydrodynamics and dynamics of nonholonomic systems [2].

Making the transformation $x = \tilde{M}^{1/2}q$, where the exponent ½ indicates the unique positive definite square root of the matrix $\tilde{M}$, and premultiplying Eq. (1.1) by $\tilde{M}^{-1/2}$ we get the following equation

$$\ddot{x} + Kx + Nx = 0 \tag{1.2}$$

where the symmetric matrix $K = \tilde{M}^{-1/2}\tilde{K}\tilde{M}^{-1/2}$ and skew-symmetric $N = \tilde{M}^{-1/2}\tilde{N}\tilde{M}^{-1/2}$. Clearly, system (1.2) is equivalent to system (1.1), and we shall from here on consider this system.

The system is said to be stable (Lyapunov stable) if every solution $x(t)$ of equation (1.2) is bounded for all non-negative t. If $N = 0$ (pure potential system), according to the famous Lagrange theorem, the system is stable if the potential matrix K is positive definite ($K > 0$); otherwise the potential system is unstable. For $N \neq 0$, it is possible that non-conservative positional forces can destabilize a stable purely potential system, and that they can stabilize an unstable potential system [1,3]. The study of the influence of circulatory forces on the stability of potential systems, including many specific problems, has a rich history (see, for example, monographs [1, 2, 4-6]).

All solutions of Eq. (1.2) can be characterized algebraically using properties of the quadratic matrix polynomial $L(\mu) = \mu^2 I + K + N$, where I is identity matrix. The eigenvalues of the system (1.2) are zeros of the characteristic polynomial $\Delta(\mu)=\det(L(\mu))$, and the multiplicity of an eigenvalue is the order of the corresponding zero in $\Delta(\mu)$. If μ is an eigenvalue, the nonzero vectors in the null space of $L(\mu)$ are the eigenvectors associated with μ. Since $\Delta(\mu)=\Delta(-\mu)$, then all eigenvalues (spectrum of the system) are located symmetrically with respect to both the real and imaginary axes in the complex plane. This means that system (1.2) is stable only when every eigenvalue μ is on the imaginary axis and simple or semi-simple, i.e., if the eigenvalue has multiplicity *k*, there are *k* linearly independent associated eigenvectors.

Although the eigenvalue analysis (spectral analysis) of the system with the help of computer programs is in principle easy, the influence of forces and parameters in the system matrices on the stability becomes lost [7]. This is more or less also the case when applying the general algebraic criteria [8,9], which are similar to the well-known Routh-Hurwitz criterion for asymptotic stability, and which are presented in Section 2. Therefore, alternative criteria, such as those that provide simpler conditions directly in terms of the matrices K and N, are more attractive, and many attempts have been made to establish such criteria. An overview of such criteria is taken from our recent paper [10], and it is given in Sections 3 and 4. In Section 5 the study of the influence of infinitesimal circulatory forces on the stability of potential systems with multiple natural frequencies is reported. Such a situation of having multiple natural frequencies can, and often does, arise in complex multi-degree-of-freedom systems such as spacecraft and building structures in which, say, the fourth bending frequency coincides with the second torsional frequency of vibration of the structure, see [11].

It should be noted that an approach to studying the stability of multi-parameter circulatory systems, related to investigating singularities on the stability boundary in the parameter space corresponding to multiple eigenvalues, is presented in the monograph by Kirillov [2] (also, see [6, 12]), but it is not the subject of this review.

2. TWO GENERAL ALGEBRAIC CRITERIA

The characteristic polynomial of Eq. (1.2) has the form

$$\Delta(\mu)=\det(L(\mu))=a_0\mu^{2n}+a_1\mu^{2(n-1)}+\ldots+a_{n-1}\mu^2+a_n, \qquad (2.1)$$

where $a_0,a_1,\ldots,a_n$ are real coefficients. It is clear that $a_0=1$. Write the $2n\times 2n$ discriminant matrix for $\Delta(\mu)$

$$\begin{pmatrix} a_0 & a_1 & a_2 & a_3 & \dots & a_n & 0 & 0 & 0 \\ 0 & n & (n-1)a_1 & (n-2)a_2 & \dots & a_{n-1} & 0 & 0 & 0 \\ 0 & a_0 & a_1 & a_2 & \dots & a_{n-1} & 0 & 0 & 0 \\ 0 & 0 & n & (n-a)a_1 & \dots & 2a_{n-2} & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 & a_0 & a_1 & \dots & a_n \\ 0 & 0 & 0 & \dots & 0 & 0 & n & \dots & a_{n-1} \end{pmatrix}. \quad (2.2)$$

The sequence

$$D_1, D_2, ..., D_n \quad (2.3)$$

where D_i is the determinant of the submatrix of (2.2) formed by the first $2i$ rows and $2i$ columns, is called a discriminant sequence of the polynomial (2.1).

Theorem 2.1 ([8]). *A necessary and sufficient condition for all the eigenvalues μ of the system (1.2) to be with zero real parts (i. e., for the system to be spectral stable) is that the elements of the discriminant sequence (2.3) are all nonnegative and that all the coefficients of the polynomial (2.1) are nonnegative.*

It is useful to distinguish between two different kinds of instability of the system (1.2):

— The system (1) is *statically unstable (divergence)* if at least one of the eigenvalues μ_i is real while remaining eigenvalues are on the imaginary axis (there is an aperiodic, exponentially growing motion);

— The system (1) is *oscillatory unstable (flutter)* if at least one of the eigenvalues μ_i is complex with non-zero real part (there is an oscillating motion with exponentially growing amplitude).

Remark 2.1. Note that violation of the conditions of Theorem 2.1 implies instability, either flutter ($D_i < 0$ for some $i \in \{1,...,n\}$) or divergence (all $D_i \geq 0$ and $a_i < 0$ for some $j \in \{1,...,n\}$) [8].

Another approach based on the properties of a quadratic form, the coefficients of which are traces of the powers of the matrix $-(K+N)$, was developed in [9].

Define the quadratic form

$$p(\xi) = \xi^T P \xi, \;\; \xi \in \Re^n, \quad (2.4)$$

with

$$P = \begin{pmatrix} n & p_1 & p_2 & \cdots & p_{n-1} \\ p_1 & p_2 & p_3 & \cdots & p_n \\ p_2 & p_3 & p_4 & \cdots & p_{n+1} \\ . & . & . & . & . \\ p_{n-1} & p_n & p_{n+1} & . & p_{2n-2} \end{pmatrix}, \tag{2.5}$$

where

$$p_k = (-1)^k Tr(K+N)^k, \ k = 1,2,...,2n-2. \tag{2.6}$$

Here and henceforth TrB stands for the trace of a matrix B.

Theorem 2. 2 ([9]). *The system (2) is:*

a) spectrally stable if and only if $p(\xi) \geq 0$*, and* $a_i \geq 0$, $i = 1,...,n$*;*

b) unstable by divergence if and only if $p(\xi) \geq 0$*, and* $\exists a_i < 0$*;*

c) unstable by flutter if and only if $p(\xi)$ *can take negative values.*

We remark that the coefficients a_i of the polynomial (2.1) can be expressed in terms of p_i by means of the Leverrier algorithm [13]

$$ka_k = -p_k - a_1 p_{k-1} - ... - a_{k-1} p_1, \ k = 1,2,...,n. \tag{2.7}$$

In the case when system (1.2) dependent on physical parameters, formulae (2.6) and (2.7) provide a connection between the conditions for the system to be spectrally stable (unstable by divergence or by flutter) and the parameters of the system. Theorem 2.2 can then be used to divide the space of parameters into the regions of spectral stability, divergence and flutter. For systems with small degrees of freedom this approach yields results straightforwardly, but it, as well as the approach based on Theorem 2.1, becomes numerically involved for larger dimensional systems.

Both the above criteria guarantee only spectral stability of the non-conservative undamped system, but not the Lyapunov stability, because among the eigenvalues on the imaginary axis there can be multiple ones, without a complete set of corresponding eigenvectors, and in this case the secular terms appear in solutions of Eq. (1.2).

Theorem 2.3 ([9]). If $p(\xi) > 0$ *and all the coefficients* a_i *of the characteristic polynomial (2.1) are positive, then the system (1.2) is Lyapunov stable*.

According to Theorem 2.2-a and Theorem 2.3, the boundaries of the spectral and Lyapunov stability of non-conservative undamped systems may be identical, as the following example shows. If the boundaries are not identical, they differ by a small set. The stability, divergence and flutter boundaries for multi-parameter non-conservative undamped systems in the generic case were investigated in [2,12].

Example 2.1. Consider the three degrees of freedom system with

$$K = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & k \\ 0 & k & 1 \end{pmatrix}, \quad N = c\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{2.8}$$

where k and c are real numbers.

From (2.6), (2.7) and (2.8), we have

$$\begin{gathered} p_1 = -3, \;\; p_2 = 3 + 2(k^2 - c^2), \\ p_3 = -3 - 6(k^2 - c^2), \\ p_4 = 3 + 12(k^2 - c^2) + 2(k^2 - c^2)^2, \end{gathered} \tag{2.9}$$

and

$$a_1 = 3, \;\; a_2 = 3 + c^2 - k^2, \;\; a_3 = 1 + c^2 - k^2. \tag{2.10}$$

In the matrix

$$P = \begin{pmatrix} 3 & p_1 & p_2 \\ p_1 & p_2 & p_3 \\ p_2 & p_3 & p_4 \end{pmatrix}, \tag{2.11}$$

the first diagonal entry is positive. Hence, by Schur's complement [14], the matrix P is positive semi-definite if and only if the two dimensional matrix

$$\hat{P} = \begin{pmatrix} p_2 - p_1^2/3 & p_3 - p_1 p_2/3 \\ p_3 - p_1 p_2/3 & p_4 - p_2^2/3 \end{pmatrix} = 2(k^2 - c^2)\begin{pmatrix} 1 & -2 \\ -2 & 4 + (k^2 - c^2)/3 \end{pmatrix} \tag{2.12}$$

is positive semi-definite, i.e., if $k^2 - c^2 \geq 0$. Then, according to Theorem 2.2, the system is: spectrally stable if $0 \leq k^2 - c^2 \leq 1$, unstable by divergence if $k^2 - c^2 > 1$, and unstable by flutter if $k^2 - c^2 < 0$ (see Fig. 2.1). If $0 < k^2 - c^2 < 1$, according to Theorem 2.3, the system is stable in the sense of Lyapunov. By means of the eigenvalue analysis, one can further show that the system is not Lyapunov stable on the boundaries $k^2 - c^2 = 1$ and $c = \pm k$, except at the point (0,0).

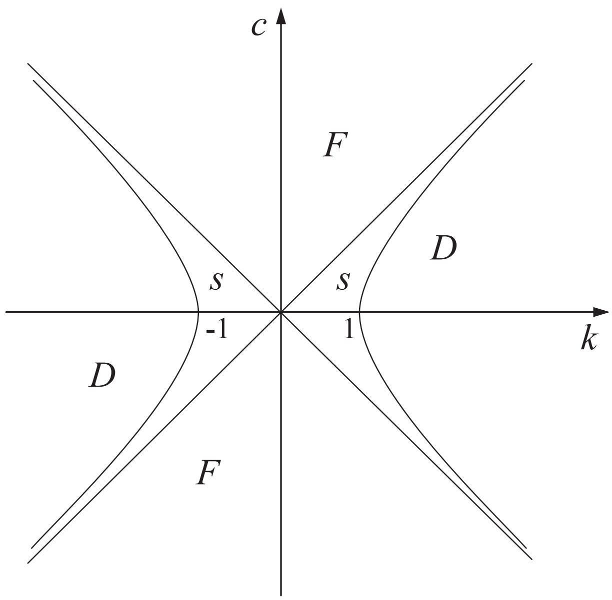

Fig. 2.1. Stability (S), divergence (D) and flutter (F) domains for the example 2.1.

3. SIMPLE STABILITY CRITERIA

There are only a few simple criteria that contain sufficient conditions for the stability of the systems under consideration.

It is frequently the case that the potential matrix is positive definite ($K > 0$). For this case, using Lyapunov's direct method, Agafonov [15,16] proved the following result.

Theorem 3.1 ([15,16]). *The system (1.2) is stable if* $K = diag(\lambda_1, \lambda_2, ..., \lambda_n)$, $\lambda_1 > \lambda_2 > ...\lambda_n > 0$, *and*

$$\| N \|_\infty < \{[(\lambda_1 + \lambda_n)^2 + 2\lambda_n s]^{1/2} - (\lambda_1 + \lambda_n)\} / 2, \tag{3.1}$$

where $s = \min_{1 \le i \ne j \le n} | \lambda_i - \lambda_j |$ *and* $\| . \|_\infty$ *denotes the maximum absolute row sum norm.*

Recently, by means of the well-known Bauer-Fike localization theorems, the following assertion was proved, which also covers the case $K > 0$.

Theorem 3.2 ([17]). *Let* $\lambda_1, ..., \lambda_n$ *be the eigenvalues of the positive definite potential matrix* K. *If*

$$\| N \|_2 < \min_{1\le i\ne j\le n} |\lambda_i - \lambda_j| / 2, \tag{3.2}$$

where $\| N \|_2$ *denotes the spectral norm of the matrix* N *(i. e., the square root of the maximum eigenvalue of* $N^T N$*) , then the system (1.2) is stable.*

It should be noted that the condition (3.2) in Theorem 3.2 can be replaced by the cruder inequality

$$\| N \|_\infty < \min_{1\le i\ne j\le n} |\lambda_i - \lambda_j| / 2, \tag{3.3}$$

which is easy to check, because $\| N \|_2 \le \| N \|_\infty$. Also, it is easy to see that whenever (3.1) is satisfied, then condition (3.3), as well as (3.2), is also satisfied.

The following numerical example shows that Theorem 3.2 significantly improves Theorem 3.1.

Example 3.1. Consider the system of three degree of freedom with

$$N = \nu \begin{pmatrix} 0 & 3 & 4 \\ -3 & 0 & 0 \\ -4 & 0 & 0 \end{pmatrix}, \ \nu \in \Re, \ K = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 8 \end{pmatrix}.$$

For this system, we calculate $\| N \|_2 = 5|\nu|$, $\| N \|_\infty = 7|\nu|$, and s = 3. The conditions (3.3) and (3.4) yield $|\nu| < 0.3$ and $|\nu| < 0.214...$, respectively, while Theorem 3.1 (condition (3.1)) predicts that the system of this example is stable if $|\nu| < 0.023...$.

When the potential matrix is not positive definite, the following condition sufficient for stability was obtained also by Agafonov [15].

Theorem 3.3 ([15]). *The system (1.2) is stable if* $K = diag(\lambda_1, \lambda_2, ..., \lambda_n)$, $\lambda_1 > \lambda_2 > ...\lambda_{n-1} > 0$, $\lambda_n \le 0$, *and*

$$a + b - [(a-b)^2 + 4s^{-2} \| N \|_\infty^4]^{1/2} > 64 s^{-3} (8\lambda_1 + s)(s - 4\| N \|_\infty)^{-1} \| N \|_\infty^4,$$

with

$$a = \lambda_{n-1} + \sum_{k=1, k\ne n-1}^{n} \nu_{n-1k}^2 / (\lambda_k - \lambda_{n-1}) \text{ and } b = \lambda_n + \sum_{k=1}^{n-1} \nu_{nk}^2 / (\lambda_k - \lambda_n),$$

where ν_{ii} *denote elements of the circulatory matrix* N.

In the case $\lambda_n = 0$, this result can be improved as follows.

Theorem 3.4 ([17]). *Let* $K = diag(\lambda_1, \lambda_2, \ldots, \lambda_n)$ *with* $\lambda_1 > \lambda_2 > \ldots > \lambda_{n-1} > 0$ *and* $\lambda_n = 0$. *If* $\| N \|_2 < \min_{1 \le i \ne j \le n} | \lambda_i - \lambda_j | / 2$ *and* $\sum_{j=1}^{n-1} | \nu_{nj} | \ne 0$, *where* ν_{ij} *are the coefficients of the matrix* N, *then the system (1.2) is stable.*

To compare Theorem 3.4 and Theorem 3.3, we consider a two degree of freedom system with

$$K = \lambda_1 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \text{ and } N = \nu \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

where $\lambda_1 > 0$ and $\nu \in \Re$. Theorem 3.4 gives $0 < 2 | \nu | \lambda_1^{-1} < 1$, which is a sufficient and necessary condition for stability of this system [1]. On the other hand, Theorem 3.3 reduces to the much stronger condition $0 < 2 | \nu | \lambda_1^{-1} < 0.103 \ldots$[15].

Suppose that

$$[K, N^2] = 0 \text{ and } [K, NKN] = 0, \tag{3.4}$$

where [.,.] denotes the commutator of two matrices. In particular, if $n = 2$, then conditions (3.4) are satisfied.

Theorem 3.5 ([18]). *If the circulatory matrix* N *is non-singular, then system (1.2), (3.4) is stable if and only if*

$$A = NKN^{-1}K - N^2 > 0 \tag{3.5}$$

and

$$K + NKN^{-1} - 2A^{1/2} > 0. \tag{3.6}$$

It should be noted that if $\det N \ne 0$, then n is necessarily even because the circulatory matrix N is skew-symmetric.

Example 3.2. Let

$$K = diag(5,5,-1,-1) \text{ and } N = \nu \begin{pmatrix} 0 & 0 & 1 & 15 \\ 0 & 0 & 15 & 1 \\ -1 & -15 & 0 & 0 \\ -15 & -1 & 0 & 0 \end{pmatrix}, \; 0 \ne \nu \in \Re. \tag{3.7}$$

Obviously, Theorems 3.1-3.4 are not applicable to this example. However, matrices (3.7) satisfy conditions (3.4) and, in addition, $\det N \neq 0$. Therefore, Theorem 3.5 is applicable. It is easy to see that conditions (3.5) and (3.6) reduce to the conditions $|\nu| > \sqrt{5}/14$ and $|\nu| < 3/16$, respectively. Thus, according to Theorem 3.5, system (1.2), (3.7) is stable only if $|\nu| \in (\sqrt{5}/14, 3/16)$.

The next criterion, related to the subclass of system (3.4) for which $K \geq 0$, is much simpler than Theorem 3.5, and it allows the possibility of $\det N = 0$ (for example, it is case when *n* is odd).

Theorem 3.6 ([17]). *a) If* $\det N \neq 0$ *and* $K \geq 0$*, then the system (1.2), (3.4) is stable if and only if*

$$[K, N]^2 - 4N^4 > 0; \tag{3.8}$$

b) If $\det N = 0$ *and* $K \geq 0$*, then the system (1.2), (3.4) is stable if and only if the following conditions are satisfied*

$$([K, N]^2 - 4N^4)|_{\mathrm{Im}\, N} > 0,\ K|_{KerN} > 0, \tag{3.9}$$

where $\mathrm{Im}\, N$ *and* $KerN$ *stand for the image and null space of* N*, respectively.*

Example 3.3. Let

$$K = diag(3,1,1) \text{ and } N = \nu \begin{pmatrix} 0 & 1 & 1 \\ -1 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix},\ \nu \in \Re. \tag{3.10}$$

The system of this example satisfies conditions (3.4), and, in addition, $\det N = 0$ and $K > 0$. Therefore, part (b) of Theorem 3.6 can be applied. This assertion yields $1 - 2\nu^2 > 0$, i. e., system (1.2), with matrices K and N as in (3.10), is stable only when $|\nu| < \sqrt{2}/2$. On the other hand, it is easy to see that the characteristic equation of this system has the following roots

$$\pm i,\ \pm i\sqrt{2 \pm \sqrt{1 - 2\nu^2}},$$

which all are purely imaginary and simple if $1 - 2\nu^2 > 0$, which is in accordance with the prediction of Theorem 3.6.

4. SIMPLE INSTABILITY CRITERIA AND GENERALIZATIONS OF THE MERKIN THEOREM

Theorem 4.1 ([4]). *The system (1.2) is unstable if* $K = 0$.

This assertion is a special case of the following result.

Theorem 4.2 ([3]). *The system (1.2) is unstable if the trace of matrix* K *is non-positive, i. e.,* $TrK \leq 0$.

Obviously, this result implies that system (1.2) is unstable if the potential matrix K is negative semi-definite (see also [19]).

We note that Theorem 4.2 is inapplicable to the case when the corresponding conservative system is stable, i. e., $K > 0$. This case is covered by the following criterion, which is easy to check.

Theorem 4.3 ([20]). *The system (1.2) is unstable if*

$$\| N \|_F^2 > \| K \|_F^2 - \frac{1}{n}(TrK)^2, \tag{4.1}$$

where $\| . \|_F$ *denotes the Frobenius norm.*

Recall that the Frobenius norm of a real matrix is defined as the square root of the sum of the squares of its elements. Condition (4.1) gives an estimation of lower bound for the intensity of circulatory forces (measured by the Frobenius norm of N) so that the introduction of arbitrary linear circulatory forces, the intensity of which is higher than this bound, into a stable potential system destroys its stability. To illustrate Theorem 4.3 we return to Example 3.3. We have $\| N \|_F^2 = 4\nu^2$, $\| K \|_F^2 = 11$, $TrK = 5$, and the instability condition (4.1) yields $|\nu| > \sqrt{2/3}$. Notice that the system of this example is unstable if and only if $|\nu| \geq \sqrt{2}/2$, as shown in the previous section.

Theorem 4.3 was obtained afterwards in [21], along with the two following sufficient conditions for the instability (also, see [9]).

Theorem 4.4 ([20]). *The system (1.2) is unstable if one of the following inequalities holds*

a). $n(\| K^2 \|_F^2 + \| N^2 \|_F^2 - 4\| KN \|_F^2 + 2Tr((KN)^2)) < (\| K \|_F^2 - \| N \|_F^2)^2$;

b) $$(\| K \|_F^2 - \| N \|_F^2)(\| K^2 \|_F^2 + \| N^2 \|_F^2 - 4\| KN \|_F^2 + 2Tr((KN)^2)) < (Tr(K^3) + 3Tr(KN^2))^2.$$

It is easy see using appropriate examples that neither Theorem 4.3 nor Theorem 4.4 implies the other one.

A remarkable result concerning the destabilizing effect of circulatory forces states: “If linear non-conservative positional forces are introduced into a stable potential system that has equal natural frequencies of vibration, then the stability will be destroyed, irrespective of any nonlinear terms” [1]. This assertion is known as Merkin’s theorem, which was first published in 1956 [4], and which can be viewed as a counterpart of one of the classical Kelvin-Tait-Chetayev stability theorems for circulatory forces (see [22]). For linear systems, this statement can be reformulated in terms of the matrices K and N as follows.

Theorem 4.5 (Merkin [1,4]). *The system (1.2) is unstable if* $N \neq 0$ *and* $K = \lambda_0 I$, $\lambda_0 \in \Re^+$.

In other words, the addition of *arbitrary* circulatory forces, infinitezimal or finite, to a consrevative system whose the potential matrix has the same eigenvalues produces instability. If we additionally assume that $\det N \neq 0$, then the system will be completely unstable (i. e., every nonzero solution $x(t)$ of Eq. (1.2) is unbounded). It follows from the following more general result which implicitly requires non-singularity of the circulatory matrix N.

Theorem 4.6 ([23]). *The system (1.2) is completely unstable if* $2N^T N - [K, N] > 0$.

Let us consider Example 3.2 again. For this system, we have

$$2N^T N - [K, N] = 2\nu \begin{pmatrix} 226\nu & 30\nu & -3 & -45 \\ 30\nu & 226\nu & -45 & -3 \\ -3 & -45 & 226\nu & 30\nu \\ -45 & -3 & 30\nu & 226\nu \end{pmatrix}.$$

It is easy to see that the above matrix is positive definite if and only if $|\nu| > 3/14$, and, according to Theorem 4.6, under this condition system (1.2), (3.7) is completely unstable.

It should be observed that the right hand side of inequality (4.1) is equal to zero only in the degenerate case when all eigenvalues of the potential matrix K are identical, and, consequently, Theorem 4.3 is a generalization of the Merkin’s theorem. Another obvious generalization is given by the following statement.

Theorem 4.7 ([18,11]). *The system (1.2) is unstable if* $N \neq 0$ *and* $[K, N] = 0$.

This generalization was obtained in [18], and independently and, more recently, in [11], where it was also pointed out that in this case the potential matrix K has at least one repeated eigenvalue (i.e., the corresponding conservative system has at least two equal natural frequencies). Although there is an uncountable infinity of skew-symmetric matrices N that commute with the given potential matrix K having multiple eigenvalues, as shown in [11], the commutation condition is very restrictive and some attempts have been made recently to weaken this restriction [24-27].

Theorem 4.8 ([24]). Let the potential matrix K has a single eigenvalue λ_0 with multiplicity $m \geq 2$, and let $T = [T_p \mid T_r]$ be an orthogonal matrix, where the $n \times p$ submatrix T_p contains any $2 \leq p \leq m$ eigenvectors (their order is immaterial) of K corresponding to the multiple eigenvalue, and the $n \times r$ submatrix T_r contains the remainder (i.e., $r = n - p$) of the eigenvectors of K. Then, if the following conditions hold

$$T_p^T N T_p \neq 0, \; T_p^T N T_r = 0, \tag{4.2}$$

the system (1.2) is unstable.

This criterion is a special case of a result that is related to general positional perturbations [24], and it also, in the case $p = m$, was obtained in [25]. It is obvious that under condition (4.2) equations (1.2) may be decoupled by using a coordinate transformation determined by the orthogonal matrix T into two subsystems, one of which assures instability. However, for the given matrix K with multiple eigenvalues the orthogonal matrix T that diagonalize K is not unique, making it difficult to apply this result. An alternative rank criterion which gives the same instability result as Theorem 4.8, but which avoids the non-uniqueness of the matrix T, as well as the calculations of its eigenvalues and eigenvectors was developed in [26].

Introduce two semi-definite matrices

$$\Omega_m = \sum_{k,l=1}^{m-1} [K^k, N^l]^T [K^k, N^l] \text{ and } \Phi = \sum_{k=0}^{n-1} K^k N^2 K^k \tag{4.3}$$

as well as the following matrices

$$S_m = \begin{bmatrix} [K,N] \\ [K,N^2] \\ \vdots \\ [K,N^{m-1}] \\ [K^2,N] \\ \vdots \\ [K^{m-1},N^{m-1}] \end{bmatrix}, \text{ and } L = \begin{bmatrix} N \\ NK \\ \vdots \\ NK^{n-1} \end{bmatrix} \tag{4.4}$$

where m is a natural number such that $m \leq n$.

Theorem 4.9 ([26]). *Let a system (1.2) be given. If*

$$rank\Phi \geq 2 + rank\Omega_w, \tag{4.5}$$

where the matrices Φ *and* Ω_w *are determined by (4.3) and* $w = rank\Phi$*, then the matrix* K *has at least one repeated eigenvalue and system (1.2) is unstable.*

In condition (4.5) the matrices Φ and Ω_w can be replaced by the matrices L and S_w, respectively, since $rank\Phi = rankL$ and $rank\Omega_w = rankS_w$.

The application of Theorem 4.8 and Theorem 4.9 is illustrated by the following example [26].

Example 4.1. Consider the system (1.2) with

$$K = \begin{bmatrix} 2 & -1 & 0 & -1 \\ -1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ -1 & 1 & 0 & 2 \end{bmatrix} \text{ and } N = \nu \begin{bmatrix} 0 & 2 & -1 & -2 \\ -2 & 0 & 2 & -1 \\ 1 & -2 & 0 & -2 \\ 2 & 1 & 2 & 0 \end{bmatrix}, \tag{4.6}$$

where ν is a nonzero real parameter.

The matrix K has the following eigenvalues and corresponding mutually orthogonal eigenvectors:

$$\lambda_{1,2,3} = 1,\ t_1^{(1)} = \frac{1}{\sqrt{3}}[1 \quad 0 \quad 1 \quad 1]^T,\ t_2^{(1)} = \frac{1}{\sqrt{3}}[1 \quad 1 \quad -1 \quad 0]^T,$$

$$t_3^{(1)} = \frac{1}{\sqrt{3}}[0 \quad 1 \quad 1 \quad -1]^T,\ \lambda_4 = 4,\ t_4^{(1)} = \frac{1}{\sqrt{3}}[1 \quad -1 \quad 0 \quad -1]^T.$$

For this system of eigenvectors, it is easy to see, the second condition of (4.2) is not satisfied. However, the following system of 4 mutually orthogonal vectors

$$t_1^{(2)} = \frac{1}{\sqrt{15}}[1 \quad -2 \quad -1 \quad 3]^T,\ t_2^{(2)} = \frac{1}{3\sqrt{15}}[9 \quad 7 \quad 1 \quad 2]^T,$$

$$t_3^{(2)} = \frac{1}{3\sqrt{3}}[0 \quad -1 \quad 5 \quad 1]^T,\ t_4^{(2)} = \frac{1}{\sqrt{3}}[1 \quad -1 \quad 0 \quad -1]^T,$$

are also eigenvectors of the matrix K associated with the eigenvalues $\lambda_{1,2,3} = 1$ and $\lambda_4 = 4$, respectively, and we have

$$\begin{bmatrix} t_1^{(2)} & t_2^{(2)} \end{bmatrix}^T N \begin{bmatrix} t_1^{(2)} & t_2^{(2)} \end{bmatrix} = 3\nu \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$

and

$$\begin{bmatrix} t_1^{(2)} & t_2^{(2)} \end{bmatrix}^T N \begin{bmatrix} t_3^{(2)} & t_4^{(2)} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Thus, for this choice of eigenvectors of the matrix K conditions (4.2) are satisfied and, according to Theorem 4.8, system (1.2), (4.6) is unstable for any nonzero value of the parameter ν. We now apply Theorem 4.9. It is clear that for this example $w = rank\Phi = 4$, since $\det N \neq 0$. Also, we have

$$\Omega_4 = \sum_{k,l=1}^{3} [K^k, N^l]^T [K^k, N^l] = 114882\nu^2 \begin{bmatrix} 9 & -9 & 0 & -9 \\ -9 & 10 & -5 & 8 \\ 0 & -5 & 25 & 5 \\ -9 & 8 & 5 & 10 \end{bmatrix},$$

which yields $rank\Omega_4 = 2$, and, according to condition (4.5), instability follows.

For the systems for which $[K, N^2] = 0$, condition (4.5) can be replaced by a simpler one as follows.

Theorem 4.10 ([26]). *If* $[K, N^2] = 0$ *and*

$$rankN \geq 2 + rank \sum_{k=1}^{r-1} [K^k, N]^T [K^k, N], \tag{4.7}$$

where $r = rankN$*, then the matrix* K *has at least one repeated eigenvalue and the system (1.2) is unstable.*

In this criterion, condition (4.7) can be replaced by the following one [26]

$$rankN \geq 2 + rank \begin{bmatrix} [K, N] \\ [K^2, N] \\ \vdots \\ [K^{r-1}, N] \end{bmatrix}.$$

For the systems confided by conditions (3.4) a much simpler criterion than the one given by Theorem 4.10 can be formulated as follows.

Theorem 4.11 ([27]). *Suppose that conditions (6) are satisfied and*

$$rank[K,N] < rankN\,, \tag{4.8}$$

then the potential matrix K *has at least one repeated eigenvalue and the system (1.2) is unstable*.

If $N \neq 0$ and the matrices K and N commute, then conditions (3.4), (4.8) obviously hold and Theorem 4.7 is a direct consequence of Theorem 4.11.

Let us now go back to Example 4.1 to illustrate the above criterion. We have $[K,N^2]=0$, $[K,NKN]=0$ (i. e., conditions (3.4) are satisfied), and

$$[K,N] = \nu\begin{pmatrix} 0 & 1 & -5 & -1 \\ 1 & -2 & 5 & 0 \\ -5 & 5 & 0 & 5 \\ -1 & 0 & 5 & 2 \end{pmatrix}.$$

Now, we calculate: $rankN = 4$ and $rank[K,N] = 2$. Thus, all conditions of Theorem 4.11 are satisfied and system (1.2), with matrices K and N as in (4.6), is unstable for every $\nu \neq 0$.

5. THE EFFECT OF INFINITESIMAL CIRCULATORY FORCES ON STABILITY OF POTENTIAL SYSTEMS

In Eq. (1.2), we substitute the circulatory matrix N with εN, where ε is a dimensionless parameter which we introduce to characterize the intensity of circulatory forces, i.e., we consider the system

$$\ddot{x} + Kx + \varepsilon Nx = 0\,, \tag{5.1}$$

where K and N are same as in (1.2). Additionally, we assume that $K > 0$, i.e. that the corresponding conservative system is stable. If all eigenvalues of the potential matrix K are distinct, according to Theorem 3.2, system (5.1) is stable for sufficiently small $|\,\varepsilon\,|$. On the other hand, if the matrices K and N satisfy the condition of Theorem 4.9, then this condition also holds when the matrix N is replaced by εN, and, consequently, system (5.1) is unstable for arbitrary nonzero $|\,\varepsilon\,|$, including infinitesimal small.

The following example clearly shows that the class of infinitesimal circulatory forces causing instability is wider than that proposed by Theorem 4.9.

Example 5.1. Let

$$K = diag(1,1,4) \text{ and } N = \begin{pmatrix} 0 & 1 & 6 \\ -1 & 0 & 0 \\ -6 & 0 & 0 \end{pmatrix}. \qquad (5.2)$$

For this example, the condition of Theorem 4.9 is not satisfied. However, it can be shown, using theorems 2.2 and 2.3, that system (5.1), (5.2) is unstable and exhibits flutter if $|\varepsilon| \in (0,a) \cup (b,\infty)$, and stable if $|\varepsilon| \in (a,b)$, where $a = 0.157...$ and $b = 0.253...$.

Therefore, the following problem arises: In system (5.1), let the positive definite potential matrix K have one multiple eigenvalue of multiplicity $m \geq 2$. Under what conditions is system (5.1) unstable (stable) for arbitrarily small nonzero $|\varepsilon|$?

To solve this problem, an approach based on classical perturbation theory for eigenvalues was recently developed [28], and the main stability results are reported below.

Suppose that the potential matrix K has one eigenvalue λ_0 of multiplicity $m \geq 2$, and that the other eigenvalues λ_i, $i = m+1,...,n$, are simple. Let $T = [T_m \,|\, T_{n-m}]$ be an orthogonal matrix, where the $n \times m$ submatrix T_m contains m eigenvectors of K corresponding to the eigenvalue λ_0, and the $n \times (n-m)$ submatrix T_{n-m} contains the remainder $n-m$ of the eigenvectors of K corresponding to the eigenvalues $\lambda_i \neq \lambda_0$, $i = m+1,\ldots,n$. The orthogonal matrix T reduces K and N to the forms

$$\hat{\Lambda} = T^T K T = diag(\lambda_0 I_m, \Lambda_{n-m}),\ \hat{N} = T^T N T = \begin{bmatrix} \hat{N}_{11} & \hat{N}_{12} \\ -\hat{N}_{12}^T & \hat{N}_{22} \end{bmatrix},$$

where the ($n-m$)-dimensional diagonal matrix $\Lambda_{n-m} = T_{n-m}^T K T_{n-m}$ contains all the eigenvalues of K that are distinct from λ_0, I_m is the m-dimensional identity matrix, and

$$\hat{N}_{11} = T_m^T N T_m = -\hat{N}_{11}^T,\ \hat{N}_{12} = T_m^T N T_{n-m},\ \hat{N}_{22} = T_{n-m}^T N T_{n-m} = -\hat{N}_{22}^T. \qquad (5.3)$$

Introduce the following $m \times m$ matrices

$$S = S^T = \hat{N}_{12} D \hat{N}_{12}^T \qquad (5.4)$$

and

$$G = -G^T = \hat{N}_{12} D \hat{N}_{22} D \hat{N}_{12}^T, \tag{5.5}$$

where the diagonal matrix $D = (\Lambda_{n-m} - \lambda_0 I_{n-m})^{-1}$.

Theorem 5.1 ([28]). *If the positive definite* potential *matrix* K *has an eigenvalue* λ_0 *of multiplicity* m, $2 \le m \le n$, *and*

$$\hat{N}_{11} = T_m^T N T_m \ne 0, \tag{5.6}$$

where the columns of the n by m matrix T_m *are orthonormal eigenvectors of* K *corresponding to the multiple eigenvalue* λ_0, *then the system described by Eq. (5.1) is unstable by flutter for arbitrarily small nonzero values of* $|\varepsilon|$.

Note that in the case $m = 2$, this result may also be established by an application of the results given in [5, Chapter 4], concerning the singularities on the stability boundary of a multi-parameter circulatory system (see also [29]).

Theorem 5.2 ([28]). *If* $\hat{N}_{11} = T_m^T N T_m = 0$ *and the symmetric matrix* S *determined by Eq. (5.4) has all distinct eigenvalues, then the system described by Eq. (5.1) is stable for sufficiently small values of* $|\varepsilon|$.

Theorem 5.3 ([28]). *If* $\hat{N}_{11} = T_m^T N T_m = 0$ *and the symmetric matrix* S *determined by Eq. (5.4) has a multiple eigenvalue* s_0 *of multiplicity* r, $2 \le r \le m$, *and* $\widetilde{T}_r^T G \widetilde{T}_r \ne 0$, *where* G *is the skew-symmetric matrix determined by Eq. (5.5) and* $\widetilde{T}_r$ *is an* $m \times r$ *matrix of the* r *orthonormal eigenvectors of* S *that belong to the eigenvalue* s_0, *then the system described by Eq. (5.1) is unstable by flutter for arbitrarily small nonzero values of* $|\varepsilon|$.

Remark 5.1. If $\breve{N}_{11} = 0$ and $\breve{N}_{12} = 0$, the system described by Eq. (5.1) is stable for sufficiently small values of $|\varepsilon|$. Indeed, in this case the system in normal coordinates is decoupled into two subsystems, one of which is an *m*-dimensional stable purely potential system independent of ε and the other is an (*n*-*m*)-dimensional circulatory system that is stable for sufficiently small $|\varepsilon|$.

Theorem 5.4 ([28]). *If* $\breve{N}_{11} = 0$ *and* $\breve{N}_{22} = 0$, *then the system described by Eq. (5.1) is stable for sufficiently small values of* $|\varepsilon|$.

Example 5.2. Consider system (5.1) with

$$K = diag(1,1,2,5) \text{ and } N = \begin{bmatrix} 0 & a & b & 0 \\ -a & 0 & 0 & 2b \\ -b & 0 & 0 & c \\ 0 & -2b & -c & 0 \end{bmatrix}, \tag{5.7}$$

where at least one of the real numbers a, b and c is nonzero.

For this system $\lambda_0 = 1$ and $m = 2$. Obviously,

$$\hat{N}_{11} = a\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

and if $a \neq 0$, in view of Theorem 5.1, for arbitrarily small nonzero values of $|\varepsilon|$, the system of this example is unstable by flutter. Now suppose $a = 0$. Then $\hat{N}_{11} = 0$, and the matrices (5.4) and (5.5) have the forms

$$S = b^2 I_2,\ G = b^2 c\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

From this follows that $s_0 = b^2$, $r = 2$ and $\widetilde{T}_2^T G \widetilde{T}_2 = G$ because $\widetilde{T}_2 = I_2$. Thus, according to Theorem 5.3, if $a = 0$ and $bc \neq 0$, the flutter instability again follows. On the other hand, if $a = 0$ and $bc = 0$, it follows from Remark 5.1 and Theorem 5.4 that the system is stable for sufficiently small values of ε. We also observe that the condition of Theorem 4.9 is satisfied if and only if $a \neq 0$ and $b = 0$, and hence under these conditions, the system (5.1), (5.7) is unstable for any nonzero ε.

Finally, we mention that for systems with less than 5 degrees of freedom, based on the above results, all skew-symmetric matrices N can be described, such that circulatory forces determined by εN, where the parameter ε is arbitrarily small, cause flutter instability in potential systems with multiple frequencies [28].

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STABILNOST NEKONZERVATIVNIH NEPRIGUŠENIH DINAMIČKIH SISTEMA: STARI I NOVI REZULTATI

Sažetak

U radu se razmatraju problemi stabilnosti linearnih nekonzervativnih neprigušenih dinamičkih sistema sa više stepeni slobode. Prikazana su dva opšta algebarska kriterijuma koji sadrže potrebne i dovoljne uslove spektralne stabilnosti, statičke i dinamičke nestabilnosti. Zatim se daje pregled jednostavnijih kriterijuma Ljapunovljeve stabilnosti i nestabilnosti formulisanih preko svojstava opisnih matrica razmatranih sistema. Posebna pažnja je posvećena nedavnim uopštenjima poznate Merkinove teoreme o nestabilnosti, kao i rezultatima istraživanja uticaja infinitezimalnih cirkulacionih sila na stabilnost potencijalnih sistema sa višestrukim prirodnim frekvencijama. Nekoliko primjera je dato radi ilustracije primjenjljivosti odabranih kriterijuma i njihovog međusobnog odnosa.