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Solving Lamé's problems by using Love's and Boussinesq's potentials

A b s t r a c t

The solutions of Lamé's problems of an axisymmetrically loaded hollow cylinder under plane strain or plain stress conditions and a pressurized spherical container are presented by using Love's and Boussinesq's potentials from the three-dimensional axisymmetric theory of elasticity. These solutions complement the classical solutions derived by the direct integration of the Navier equations of equilibrium, or by using the Airy stress function of the two-dimensional theory of elasticity. An advantage of the solutions based on the use of Love's and Boussinesq's potentials is that the displacement and the stress components can both be expressed explicitly in terms of these potentials. The stress and displacement fields in a hollow disk subjected to distributed bending moments along its circular boundaries are also derived.

Keywords: Airy stress function; Boussinesq's potentials; circular plate; Lamé's problem; Love's potential; plane strain; spherical container

1 Introduction

The Lamé problem of a pressurized hollow cylinder under conditions of plane strain or plane stress is one of the classical problems of the theory of elasticity (e.g., Sokolnikoff, 1956; Timoshenko and Goodier, 1970; Lurie, 2005; Sadd, 2014). Its solution is commonly derived by the direct integration of the Navier equations of equilibrium, or by using the Airy stress function of the two-dimensional theory of elasticity. In the former approach, the non-trivially satisfied Navier equations are

$$\frac{\mathrm{d}^2 u_r}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}u_r}{\mathrm{d}r} - \frac{u_r}{r^2} = 0\,, \quad \frac{\mathrm{d}^2 u_z}{\mathrm{d}z^2} = 0\,, \tag{1.1}$$

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where $u_r = u_r(r)$ is the radial component of the displacement, and $u_z = u_z(z)$ is the out-of-plane displacement component. The integration of these two differential equations gives

$$u_r = c_1 r + \frac{c_2}{r}\,, \quad u_z = cz\,. \tag{1.2}$$

The integration constants c_1 and c_2 can be determined by applying the boundary conditions at the inner and outer boundary $r = a$ and $r = b$, which may be given either in terms of the prescribed radial traction or prescribed radial displacement. The constant c is equal to zero for plane strain, and is different from zero in the case of plane stress, being determined from the condition $\sigma_{zz} = 0$.

In the approach based on the Airy stress function $\phi = \phi(r)$, the in-plane stress components are expressed as

$$\sigma_{rr} = \frac{1}{r}\frac{\mathrm{d}\phi}{\mathrm{d}r}\,, \quad \sigma_{\theta\theta} = \frac{\mathrm{d}^2\phi}{\mathrm{d}r^2}\,, \tag{1.3}$$

where (r, θ) are the polar coordinates within the plane of the cross section of the cylinder, orthogonal to its thickness. The Airy stress function satisfies the biharmonic equation $\nabla^4\phi = 0$. For the Lamé problem, by inspection, ϕ is of the form (e.g., Malvern, 1969)

$$\phi = \frac{1}{2}\,k_1 r^2 + k_2 \ln r\,. \tag{1.4}$$

The corresponding stress components are, from (1.3),

$$\sigma_{rr} = k_1 + \frac{k_2}{r^2}\,, \quad \sigma_{\theta\theta} = k_1 - \frac{k_2}{r^2}\,. \tag{1.5}$$

The constants k_1 and k_2 follow from the boundary conditions at $r = a$ and $r = b$. The displacements cannot be expressed explicitly in terms of ϕ, but can be obtained by the integration of the strain expressions $\epsilon_{rr} = \mathrm{d}u_r/\mathrm{d}r$ and $\epsilon_{\theta\theta} = u_r/r$, after they are determined from the stresses by using Hooke's law.

In the case of a pressurized spherical container, the only equation of equilibrium is in the radial direction ρ of the spherical (ρ, θ, ϕ) coordinate system,

$$\frac{\mathrm{d}^2 u_\rho}{\mathrm{d}\rho^2} + \frac{2}{\rho}\frac{\mathrm{d}u_\rho}{\mathrm{d}\rho} - 2\,\frac{u_\rho}{\rho^2} = 0\,, \quad \rho^2 = r^2 + z^2\,, \tag{1.6}$$

whose solution is

$$u_\rho = c_1\rho + \frac{c_2}{\rho^2}\,. \tag{1.7}$$

In section 2, we present the solution of the Lamé problem by using the biharmonic Love's potential function $\Omega = \Omega(r)$ of the three-dimensional axisymmetric theory of elasticity, which allows the explicit representation of both the stress and displacement components in terms of Ω. In section 3 we derive the expressions for the harmonic Boussinesq's potentials B and β, in terms of which the stress and displacement components can also be expressed explicitly. The pressurized spherical container is considered in section 4. In section 5 we derive the stress and displacement fields in a hollow disk subjected to distributed bending moments along its circular boundaries.

2 Solution of Lamé problem by Love's potential

For the three-dimensional axisymmetric elasticity problems (Love, 1944; Timoshenko and Goodier, 1970), the displacement components can be expressed in terms of Love's potential $\Omega = \Omega(r, z)$ as

$$2\mu u_r = -\frac{\partial^2\Omega}{\partial r\partial z}, \quad 2\mu u_z = 2(1-\nu)\nabla^2\Omega - \frac{\partial^2\Omega}{\partial z^2}\,. \tag{2.1}$$

It can be readily verified that (2.1) identically satisfies the axisymmetric Navier equations

$$\mu\left(\nabla^2 u_r - \frac{u_r}{r^2}\right) + (\lambda+\mu)\frac{\partial e}{\partial r} = 0\,, \quad \mu\nabla^2 u_z + (\lambda+\mu)\frac{\partial e}{\partial z} = 0\,, \tag{2.2}$$

provided that

$$\nabla^4\Omega = 0\,, \quad \nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}\,. \tag{2.3}$$

The body forces are assumed to be absent and the Lamé constants of elasticity are (λ, μ), where μ is the shear modulus. The corresponding stress components can be expressed in terms of Ω as

$$\begin{aligned} \sigma_{rr} &= \frac{\partial}{\partial z}\left(\nu\nabla^2\Omega - \frac{\partial^2\Omega}{\partial r^2}\right), \quad \sigma_{\theta\theta} = \frac{\partial}{\partial z}\left(\nu\nabla^2\Omega - \frac{1}{r}\frac{\partial\Omega}{\partial r}\right), \\ \sigma_{zz} &= \frac{\partial}{\partial z}\left[(2-\nu)\nabla^2\Omega - \frac{\partial^2\Omega}{\partial z^2}\right], \quad \sigma_{rz} = \frac{\partial}{\partial r}\left[(1-\nu)\nabla^2\Omega - \frac{\partial^2\Omega}{\partial z^2}\right]. \end{aligned} \tag{2.4}$$

The structure of the compatibility equations for three-dimensional axisymmetric problems has been recently discussed by Lubarda and Lubarda (2020a).

We now use expressions (2.1) and (2.4) to solve the Lamé problem in the case of plane strain. The solution for the in-plane stresses and the displacement component u_r in the case of plane stress can be generated from the plane strain solution by using the change of Poisson's ratio $\nu \to \nu/(1+\nu)$, while keeping the shear modulus unchanged ($\mu \to \mu$), e.g., Lubarda and Lubarda (2020b). For the plane strain case, we have $\epsilon_{zz} = 0$ and $u_z = 0$. Thus, the second expression in (2.1) gives

$$2\mu u_z = 2(1-\nu)\nabla^2\Omega - \frac{\partial^2\Omega}{\partial z^2} = 0\,. \tag{2.5}$$

This suggests that Love's potential is of the following form

$$\Omega = z\omega(r) + \frac{1}{6}c_1 z^3\,. \tag{2.6}$$

Indeed,

$$\frac{\partial^2\Omega}{\partial z^2} = c_1 z\,, \quad \nabla^2\Omega = (c_1 + \nabla^2\omega)z\,, \tag{2.7}$$

and the substitution of (2.7) into (2.5) gives the differential equation for the introduced auxiliary function $\omega = \omega(r)$,

$$\nabla^2\omega = -\frac{(1-2\nu)c_1}{2(1-\nu)}\,, \quad \nabla^2 = \frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\,. \tag{2.8}$$

Up to an immaterial constant, the solution of this differential equation is

$$\omega = -\frac{(1-2\nu)c_1}{8(1-\nu)}r^2 + c_2\ln r\,. \tag{2.9}$$

Consequently, Love's potential (2.6) is

$$\Omega = \left[-\frac{(1-2\nu)c_1}{8(1-\nu)}r^2 + c_2\ln r\right]z + \frac{1}{6}c_1 z^3\,. \tag{2.10}$$

The radial displacement is obtained by substituting (2.10) into (2.1), which gives

$$2\mu u_r = \frac{(1-2\nu)c_1}{4(1-\nu)}r - \frac{c_2}{r}\,. \tag{2.11}$$

Similarly, the in-plane stress components follow by substituting (2.10) into (2.4),

$$\sigma_{rr} = \frac{c_1}{4(1-\nu)} + \frac{c_2}{r^2}\,, \quad \sigma_{\theta\theta} = \frac{c_1}{4(1-\nu)} - \frac{c_2}{r^2}\,. \tag{2.12}$$

The out-of-plane stress component is

$$\sigma_{zz} = \nu(\sigma_{rr} + \sigma_{\theta\theta}) = \frac{\nu c_1}{2(1-\nu)}\,. \tag{2.13}$$

It is noted that Love's potential Ω in (2.10) is related to the Airy stress function ϕ from (1.4) by

$$\Omega = z\phi + \frac{c_1}{12}\, z(2z^2 - 3r^2)\,, \tag{2.14}$$

because the two sets of constants (k_1, k_2) and (c_1, c_2), appearing in (1.4) and (2.10), are related by $c_2 = k_2$ and $c_1 = 4(1-\nu)k_1$. The second term on the right-hand side of (2.14), i.e., $c_1 z(2z^2-3r^2)/12$, is a harmonic function. Physically, this term gives rise to a uniform state of stress $\sigma_{rr} = \sigma_{\theta\theta} = c_1/2$ and $\sigma_{zz} = -c_1$, with the corresponding displacement components $u_r = c_1 r/(4\mu)$ and $u_z = -c_1 z/(2\mu)$.

2.1 Examples

Figure 1 shows a hollow circular cylinder of length L whose ends are constrained such that $u_z = 0$. The outer boundary of the cylinder $r = b$ is subjected to pressure p, while the inner boundary $r = a$ is in contact with a smooth rigid inclusion which prevents radial displacement, $u_r(a) = 0$. Thus,

$$\begin{aligned} u_r(r=a) = 0: &\quad \frac{(1-2\nu)c_1}{4(1-\nu)}\, a - \frac{c_2}{a} = 0\,, \\ \sigma_{rr}(r=b) = -p: &\quad \frac{c_1}{4(1-\nu)} + \frac{c_2}{b^2} = -p\,, \end{aligned} \tag{2.15}$$

which gives the following expressions for the constants c_1 and c_2,

$$c_1 = -\frac{4(1-\nu)p}{1+(1-2\nu)a^2/b^2}\,, \quad c_2 = -\frac{(1-2\nu)a^2 p}{1+(1-2\nu)a^2/b^2}\,. \tag{2.16}$$

Thus, the displacement and stress expressions become

$$2\mu u_r = -\frac{(1-2\nu)p}{1+(1-2\nu)a^2/b^2}\left(r - \frac{a^2}{r}\right), \quad u_z = 0\,, \tag{2.17}$$

and

$$\begin{aligned} \sigma_{rr} &= -\frac{p}{1+(1-2\nu)a^2/b^2}\left[1 + (1-2\nu)\,\frac{a^2}{r^2}\right], \\ \sigma_{\theta\theta} &= -\frac{p}{1+(1-2\nu)a^2/b^2}\left[1 - (1-2\nu)\,\frac{a^2}{r^2}\right]. \end{aligned} \tag{2.18}$$

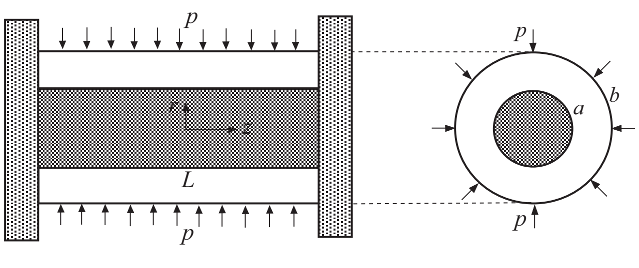

Figure 1: A hollow circular cylinder of length L whose ends are constrained by rigid vertical panels which prevent longitudinal deformation. The outer boundary of the cylinder $(r = b)$ is subjected to pressure p, while the inner boundary $(r = a)$ is in contact with a smooth rigid inclusion which prevents radial displacement.

The out-of-plane stress is

$$\sigma_{zz} = -\frac{2\nu p}{1 + (1 - 2\nu)a^2/b^2}\,. \tag{2.19}$$

For a pressurized solid cylinder ($a = 0$), the constants are $c_1 = -4(1 - \nu)p$ and $c_2 = 0$, and Love's potential (2.10) reduces to

$$\Omega = \frac{1}{2}\,pz\left[(1 - 2\nu)r^2 - \frac{4}{3}\,(1 - \nu)z^2\right]. \tag{2.20}$$

The corresponding stresses are $\sigma_{rr} = \sigma_{\theta\theta} = -p$ and $\sigma_{zz} = -2\nu p$, with the radial displacement $u_r = -(1 - 2\nu)pr/(2\mu)$.

2.1.1 Plane stress case

If ν in (2.17) and (2.18) is replaced by $\nu/(1+\nu)$, we obtain the plane stress version of the expressions (Fig. 2), i.e.,

$$2\mu u_r = -\frac{(1 - \nu)p}{1 + \nu + (1 - \nu)a^2/b^2}\left(r - \frac{a^2}{r}\right), \tag{2.21}$$

and

$$\begin{aligned} \sigma_{rr} &= -\frac{p}{1 + \nu + (1 - \nu)a^2/b^2}\left[1 + \nu + (1 - \nu)\,\frac{a^2}{r^2}\right], \\ \sigma_{\theta\theta} &= -\frac{p}{1 + \nu + (1 - \nu)a^2/b^2}\left[1 + \nu - (1 - \nu)\,\frac{a^2}{r^2}\right]. \end{aligned} \tag{2.22}$$

In the plane stress case $\sigma_{zz} = 0$, while

$$\epsilon_{zz} = -\frac{\nu}{2\mu(1 + \nu)}\,(\sigma_{rr} + \sigma_{\theta\theta}) = \frac{\nu p/\mu}{1 + \nu + (1 - \nu)a^2/b^2}\,. \tag{2.23}$$

Thus, since $\epsilon_{zz} = \mathrm{d}u_z/\mathrm{d}z$, the integration gives

$$u_z = \frac{\nu p/\mu}{1 + \nu + (1 - \nu)a^2/b^2}\,z\,, \tag{2.24}$$

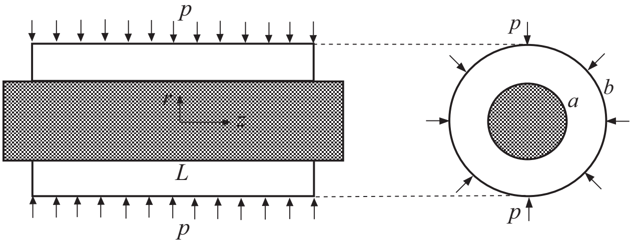

Figure 2: A hollow circular cylinder of length L whose ends $z = \pm L/2$ are traction-free. The outer boundary of the cylinder ($r = b$) is subjected to pressure p, while the inner boundary ($r = a$) is in contact with a smooth rigid inclusion which prevents radial displacement.

where the coordinate origin is in the middle cross section of the cylinder, and it is assumed that $u_z(0) = 0$. If $L \ll b$, we have the usual thin-disk configuration, although neither the radial displacement u_r in (2.21), nor the stress components σ_{rr} and $\sigma_{\theta\theta}$ in (2.22) depend on L.

In retrospect, it can be shown that Love's potential for the plane stress Lamé problem is

$$\Omega = \left[-\frac{(1-\nu)\bar{c}_1}{4(2-\nu)}\, r^2 + \bar{c}_2 \ln r \right] z + \frac{1}{6}\, \bar{c}_1 z^3 \,, \tag{2.25}$$

which complements the plane strain version (2.10). The integration constants for the problem in Fig. 2 are

$$\bar{c}_1 = -\frac{2(2-\nu)p}{1+\nu+(1-\nu)a^2/b^2}\,, \quad \bar{c}_2 = -\frac{(1-\nu)pa^2}{1+\nu+(1-\nu)a^2/b^2}\,. \tag{2.26}$$

The solution of the Lamé problem for more general axially symmetric boundary conditions, which encompass all possible combinations of kinematic and kinetic conditions at the inner and outer boundaries of a hollow cylinder, is presented by Lubarda (2009).

2.1.2 Pressurized vertical cylindrical hole in a half-space

Figure 3 shows the half-space $z \geq 0$, with an infinitely extended vertical circular cylindrical hole under internal pressure p. The boundary $z = 0$ is traction-free. By placing the coordinate origin at the center of the hole, Love's potential can be expressed as

$$\Omega = kz \ln r \,. \tag{2.27}$$

The constant k can be determined from the condition $\sigma_{rr}(r=a) = -p$, where a is the radius of the hole. This gives $k = -pa^2$. The corresponding stresses are, from (2.4),

$$\sigma_{rr} = -p\,\frac{a^2}{r^2}\,, \quad \sigma_{\theta\theta} = p\,\frac{a^2}{r^2}\,, \quad \sigma_{zz} = 0\,. \tag{2.28}$$

The stress state is deviatoric and no volume change arises at any point of the half-space. The displacement components are, from (2.3),

$$2\mu u_r = -\frac{k}{r} = p\,\frac{a^2}{r}\,, \quad u_z = 0\,. \tag{2.29}$$

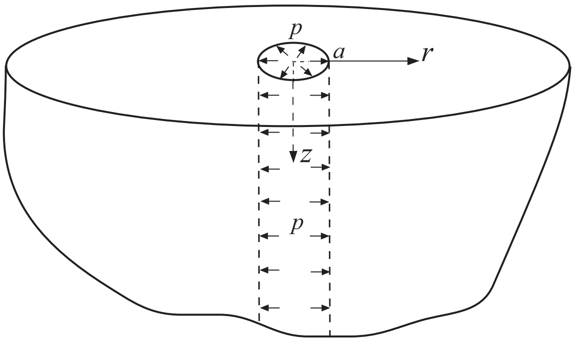

Figure 3: The half-space $z \geq 0$, with an infinitely extended vertical cylindrical circular hole under internal pressure p. The boundary $z = 0$ of the half-space is traction-free.

The derived solution can also be deduced from (2.11) and (2.12) in the limit $b \to \infty$. Because the displacement u_r and the stresses σ_{rr} and $\sigma_{\theta\theta}$ must vanish as $r \to \infty$, it follows that, in (2.11) and (2.12), $c_1 = 0$ and $c_2 = k = -pa^2$. If the Airy stress function from section 1 was used, then $\phi = k \ln r$ and, therefore, in this case $\Omega = z\phi$.

An analysis of three-dimensional stress concentration around a vertical cylindrical hole in the half-space under remote loading parallel to the plane boundary of the half-space has been presented by Youngdahl and Sternberg (1966). If the depth of the hole $L \gg a$ is finite, the elastic fields can be determined numerically by using the finite element method (e.g., Hughes, 2000; Reddy, 2005), although sufficiently away from the end of the hole ($z = L$), the stresses can be evaluated approximately by other means. For example, for $z \gg L$ (deep below the hole), the stresses can be determined from Mindlin's (1936) solution corresponding to the vertical concentrated force of magnitude $P = \pi a^2 p$. An analysis of circular inclusions in concentric cylinders by using Papkovich–Neuber potentials was presented by Lubarda (1998) and Lubarda and Markenscoff (1999).

3 Solution of Lamé's problem by Boussinesq's potentials

The displacement components can also be expressed in terms of Boussinesq's potentials B and β, which are defined such that (e.g., Lubarda and Lubarda, 2020)

$$2\mu u_r = -\frac{\partial A}{\partial r}\,, \quad 2\mu u_z = B - \frac{\partial A}{\partial z}\,, \tag{3.1}$$

where

$$A = \beta + \frac{zB}{4(1-\nu)}\,. \tag{3.2}$$

In the absence of body force, the functions B and β are harmonic ($\nabla^2 B = 0$ and $\nabla^2\beta = 0$). The auxiliary function A itself is biharmonic, because zB is biharmonic. The corresponding stress expressions are

$$\begin{aligned}
\sigma_{rr} &= -\frac{\partial^2 A}{\partial r^2} + \frac{\nu}{2(1-\nu)}\,\frac{\partial B}{\partial z}\,,\\
\sigma_{\theta\theta} &= -\frac{1}{r}\,\frac{\partial A}{\partial r} + \frac{\nu}{2(1-\nu)}\,\frac{\partial B}{\partial z}\,,\\
\sigma_{zz} &= -\frac{\partial^2 A}{\partial z^2} + \frac{2-\nu}{2(1-\nu)}\,\frac{\partial B}{\partial z}\,.
\end{aligned} \tag{3.3}$$

The stress components σ_{rr} and $\sigma_{\theta\theta}$ in the Lamé problem are reasonably expected to be independent of z, which means that the potential B should be linear in z. Thus, we assume

$$B = b_0 z\,. \tag{3.4}$$

Furthermore, for the plane strain $2\mu u_z = B - \partial A/\partial z = 0$, which implies that the function A must be quadratic in z. Consequently, it follows from (3.2) that β must be quadratic in z, as well. Therefore, we assume that

$$\beta = b_1\left(z^2 - \frac{1}{2}\,r^2\right) + b_2 \ln r\,, \tag{3.5}$$

which is clearly harmonic ($\nabla^2\beta = 0$), because $z^2 - r^2/2$ and $\ln r$ are both harmonic functions.

The substitution of (3.4) and (3.5) into (3.1) now gives

$$2\mu u_r = b_1\,r - \frac{b_2}{r}\,, \quad 2\mu u_z = \frac{1-2\nu}{2(1-\nu)}\,b_0 - 2b_1 = 0\,. \tag{3.6}$$

From the second expression in (3.6) it follows that

$$b_0 = \frac{4(1-\nu)}{1-2\nu}\,b_1\,. \tag{3.7}$$

By using this and by substituting (3.4) and (3.5) into (3.3), the stress components are found to be

$$\sigma_{rr} = \frac{b_1}{1-2\nu} - \frac{b_2}{r^2}\,, \quad \sigma_{\theta\theta} = \frac{b_1}{1-2\nu} + \frac{b_2}{r^2}\,, \quad \sigma_{zz} = \frac{2(2-\nu)b_1}{1-2\nu}\,. \tag{3.8}$$

The constants b_1 and b_2 can be determined from the specified boundary conditions, as discussed in the previous section.

3.1 Boussinesq's potentials in terms of Love's potential

Boussinesq's potentials B and β are related to Love's potential Ω by (e.g., Sadd, 2014; Lubarda and Lubarda, 2020)

$$B = 2(1-\nu)\nabla^2\Omega\,, \quad \beta = \frac{\partial\Omega}{\partial z} - \frac{1}{2}\,z\nabla^2\Omega\,. \tag{3.9}$$

Consequently, by substituting (2.10) into (3.9), we obtain in the case of plane strain

$$B = c_1 z\,, \quad \beta = \frac{(1-2\nu)c_1}{4(1-\nu)}\left(z^2 - \frac{1}{2}\,r^2\right) + c_2 \ln r\,. \tag{3.10}$$

The comparison of (3.10) with (3.4) and (3.5) establishes the relationship between the two sets of constants,

$$b_0 = c_1\,, \quad b_1 = \frac{(1-2\nu)c_1}{4(1-\nu)}\,, \quad b_2 = c_2\,. \tag{3.11}$$

In the case of plane stress, the substitution of (2.25) into (3.9) gives

$$B = \bar{b}_0 z\,, \quad \beta = \bar{b}_1\left(z^2 - \frac{1}{2}\,r^2\right) + \bar{b}_2 \ln r\,, \tag{3.12}$$

where

$$\bar{b}_0 = \frac{2(1-\nu)\bar{c}_1}{2-\nu}\,, \quad \bar{b}_1 = \frac{(1-\nu)\bar{c}_1}{2(2-\nu)}\,, \quad \bar{b}_2 = \bar{c}_2\,. \tag{3.13}$$

Finally, it may be noted that the constants (b_0, b_1, b_2) are related to the constants (k_1, k_2), appearing in the Airy stress function (1.4), by

$$b_0 = 4(1-\nu)k_1\,, \quad b_1 = (1-2\nu)k_1\,, \quad b_2 = k_2\,, \tag{3.14}$$

with similar relationships existing between the constants $(\bar{b}_0, \bar{b}_1, \bar{b}_2)$ and (k_1, k_2).

4 Pressurized spherical container

To derive Love's and Boussinesq's potentials for a pressurized spherical container of inner radius a and outer radius b, we use the superposition of the results for a pressurized spherical hole in an infinite medium and for the state of uniform hydrostatic pressure in an infinite medium. Love's potential for the problem of a pressurized spherical hole in an infinite medium can be recognized from the well-known results for the center of dilatation in an infinite medium (e.g., Love, 1944; Timoshenko and Goodier, 1970). This gives

$$\Omega = \frac{1}{2}\,p_* a^3 \ln(\rho + z)\,, \quad \rho^2 = r^2 + z^2, \tag{4.1}$$

where p_* is the applied pressure over the boundary of the hole of radius a. The corresponding stresses are

$$\begin{aligned} \sigma_{rr} &= p_* a^3\,\frac{z^2 - 2r^2}{2\rho^5}\,, \quad \sigma_{\theta\theta} = p_* a^3\,\frac{1}{2\rho^3}\,,\\ \sigma_{zz} &= p_* a^3\,\frac{r^2 - 2z^2}{2\rho^5}\,, \quad \sigma_{zr} = -3p_* a^3\,\frac{rz}{2\rho^5}\,. \end{aligned} \tag{4.2}$$

Love's potential for the state of uniform hydrostatic pressure p in the entire infinite medium (without the hole) is

$$\Omega = \frac{pz}{2(1+\nu)}\left[(1-2\nu)r^2 + \frac{1}{3}(4\nu-5)z^2\right], \tag{4.3}$$

as can be verified by evaluating the stress components and showing that indeed $\sigma_{rr} = \sigma_{\theta\theta} = \sigma_{zz} = -p$ and $\sigma_{zr} = 0$.

To solve the pressurized spherical container problem, we add (4.1) and (4.3) to obtain

$$\Omega = \frac{1}{2}\,p_* a^3 \ln(\rho + z) + \frac{pz}{2(1+\nu)}\left[(1-2\nu)r^2 + \frac{1}{3}(4\nu-5)z^2\right], \tag{4.4}$$

and determine p_0 and p by imposing the boundary conditions at $\rho = a$ and $\rho = b$.

The cylindrical stress components associated with (4.4) are

$$\begin{aligned}\sigma_{rr} &= -p + p_* a^3\,\frac{z^2-2r^2}{2\rho^5}\,, \quad \sigma_{\theta\theta} = -p + p_* a^3\,\frac{1}{2\rho^3}\,,\\ \sigma_{zz} &= -p + p_* a^3\,\frac{r^2-2z^2}{2\rho^5}\,, \quad \sigma_{zr} = -3p_* a^3\,\frac{rz}{2\rho^5}\,.\end{aligned} \tag{4.5}$$

The corresponding spherical stress components are easily recognized to be

$$\sigma_{\rho\rho} = -p - p_*\,\frac{a^3}{\rho^3}\,, \quad \sigma_{\theta\theta} = \sigma_{\phi\phi} = -p + p_*\,\frac{a^3}{2\rho^3}\,. \tag{4.6}$$

For example, if the boundary conditions are $\sigma_{\rho\rho}(r = a) = -p_{\rm i}$ and $\sigma_{\rho\rho}(r = b) = -p_{\rm o}$, we have

$$p + p_* = p_{\rm i}\,, \quad p + p_*\,\frac{a^3}{b^3} = p_{\rm o}\,. \tag{4.7}$$

This gives

$$p = \frac{p_{\rm o} b^3 - p_{\rm i} a^3}{b^3 - a^3}\,, \quad p_* = -\frac{(p_{\rm o} - p_{\rm i})b^3}{b^3 - a^3}\,. \tag{4.8}$$

4.1 Boussinesq's potentials

Boussinesq's potentials for a pressurized spherical hole in an infinite medium (e.g., Lubarda and Lubarda, 2020) are

$$B = 0\,, \quad \beta = \frac{p_* a^3}{2\rho}\,. \tag{4.9}$$

Boussinesq's potentials for an infinite medium under a uniform hydrostatic state of compressive stress p are

$$B = -\frac{6(1-\nu)}{1+\nu}\,pz\,, \quad \beta = \frac{1-2\nu}{2(1+\nu)}\,p(r^2 - 2z^2)\,. \tag{4.10}$$

Consequently, the overall Boussinesq's potentials for a pressurized spherical container are

$$B = -\frac{6(1-\nu)}{1+\nu}\,pz\,, \quad \beta = \frac{1-2\nu}{2(1+\nu)}\,p(r^2 - 2z^2) + \frac{p_* a^3}{2\rho}\,, \tag{4.11}$$

where p and p_* are determined from the boundary conditions at the inner and outer boundary of the spherical container.

5 Hollow circular plate under distributed edge bending moments

Fig. 4 shows a hollow circular disk subjected to distributed bending moments M_a and M_b (per unit length) along the boundaries $r = a$ and $r = b$, while the flat sides $z = \pm h/2$ are traction-free. The thickness of the disk is denoted by h, and the coordinate origin is at the center od the mid-plane of the disk. Using a semi-inverse method of elasticity, we reasonably assume that the radial stress varies linearly with z, i.e., $\sigma_{rr} = h(r)z$, while σ_{zz} and σ_{zr} identically vanish. Thus, from (2.4), we can write

$$\begin{aligned} \nu\nabla^2\Omega - \frac{\partial^2\Omega}{\partial r^2} &= \frac{1}{2}\,h(r)z^2\,, \\ (2-\nu)\nabla^2\Omega - \frac{\partial^2\Omega}{\partial z^2} &= f(r)\,, \\ (1-\nu)\nabla^2\Omega - \frac{\partial^2\Omega}{\partial z^2} &= g(z)\,, \end{aligned} \tag{5.1}$$

where the functions $h(r)$, $f(r)$ and $g(z)$ will be determined in the sequel. By subtracting the third from the second expression in (5.1), we obtain

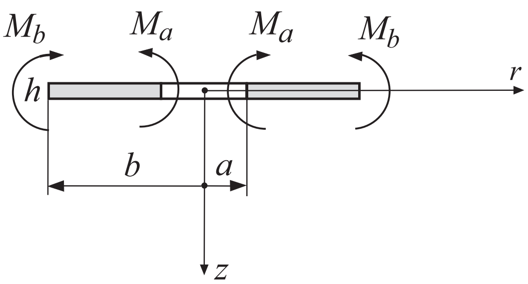

Figure 4: A hollow circular disk of thickness h subjected to distributed bending moments M_a and M_b (per unit length) along the boundary edges $r = a$ and $r = b$. The flat faces od the disk $z = \pm h/2$ are traction-free.

$$\nabla^2\Omega = f(r) - g(z)\,. \tag{5.2}$$

Applying the Laplacian ∇^2 to (5.2) and imposing the condition $\nabla^4\Omega = 0$, it then follows that

$$\frac{\mathrm{d}^2 f}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}f}{\mathrm{d}r} - \frac{\mathrm{d}^2 g}{\mathrm{d}z^2} = 0 \quad\Rightarrow\quad \frac{\mathrm{d}^2 f}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}f}{\mathrm{d}r} = \frac{\mathrm{d}^2 g}{\mathrm{d}z^2} = c_1 = \text{const.} \tag{5.3}$$

Thus, upon integration,

$$f(r) = \frac{1}{4}\,c_1 r^2 + c_2 \ln r + c_3\,, \quad g(z) = \frac{1}{2}\,c_1 z^2 + c_4 z + c_5\,. \tag{5.4}$$

To satisfy the boundary conditions, it turns out that the constant and linear terms in (5.4) are not needed, thus we take $c_3 = c_4 = c_5 = 0$. Consequently,

$$f(r) = \frac{1}{4}\,c_1 r^2 + c_2 \ln r\,, \quad g(z) = \frac{1}{2}\,c_1 z^2\,, \quad \nabla^2\Omega = \frac{1}{4}\,c_1(r^2 - 2z^2) + c_2 \ln r\,. \tag{5.5}$$

The expression for Ω can be derived by integrating

$$\frac{\partial^2\Omega}{\partial z^2} = (1-\nu)f(r) - (2-\nu)g(z)\,, \tag{5.6}$$

which follows from either second or third expression in (5.1) and expression (5.2). Upon using (5.5), it follows that

$$\Omega = \frac{1}{2}(1-\nu)\left(\frac{1}{4}c_1 r^2 + c_2 \ln r\right) z^2 - \frac{1}{24}(2-\nu)c_1 z^4 + \frac{\nu}{4}\left[\frac{1}{16}c_1 r^4 + c_2(r^2 \ln r - r^2)\right]. \tag{5.7}$$

The nonvanishing stresses σ_{rr} and $\sigma_{\theta\theta}$ are obtained by substituting (5.7) into (2.4), i.e.,

$$\sigma_{rr} = \frac{\partial}{\partial z}\left(\nu\nabla^2\Omega - \frac{\partial^2\Omega}{\partial r^2}\right), \quad \sigma_{\theta\theta} = \frac{\partial}{\partial z}\left(\nu\nabla^2\Omega - \frac{1}{r}\frac{\partial\Omega}{\partial r}\right). \tag{5.8}$$

This gives

$$\sigma_{rr} = \left[-\frac{1}{2}(1+\nu)c_1 + (1-\nu)\frac{c_2}{r^2}\right] z, \quad \sigma_{\theta\theta} = \left[-\frac{1}{2}(1+\nu)c_1 - (1-\nu)\frac{c_2}{r^2}\right] z. \tag{5.9}$$

Thus, the function $h(r)$ appearing in (5.1) is

$$h(r) = -\frac{1}{2}(1+\nu)c_1 + (1-\nu)\frac{c_2}{r^2}. \tag{5.10}$$

The bending moment per unit length around the circumference of an arbitrary radius $a \le r \le b$ is

$$M(r) = \int_{-h/2}^{h/2} \sigma_{rr} z \, \mathrm{d}z = \left[-\frac{1}{2}(1+\nu)c_1 + (1-\nu)\frac{c_2}{r^2}\right]\frac{h^3}{12}. \tag{5.11}$$

The integration constants c_1 and c_2 follow from the boundary conditions $M(a) = M_a$ and $M(b) = M_b$, i.e.,

$$-\frac{1}{2}(1+\nu)c_1 + (1-\nu)\frac{c_2}{a^2} = \frac{12M_a}{h^3}, \quad -\frac{1}{2}(1+\nu)c_1 + (1-\nu)\frac{c_2}{b^2} = \frac{12M_b}{h^3}. \tag{5.12}$$

Upon solving for c_1 and c_2, we obtain

$$c_1 = \frac{24}{(1+\nu)h^3}\,\frac{a^2 M_a - b^2 M_b}{b^2 - a^2}, \quad c_2 = \frac{12}{(1-\nu)h^3}\,\frac{a^2 b^2 (M_a - M_b)}{b^2 - a^2}. \tag{5.13}$$

By substituting (5.13) into (5.9) it follows that stresses are independent of ν and given by

$$\begin{aligned} \sigma_{rr} &= \frac{12}{h^3(b^2-a^2)}\left[b^2 M_b - a^2 M_a + (M_b - M_a)\frac{a^2 b^2}{r^2}\right] z, \\ \sigma_{\theta\theta} &= \frac{12}{h^3(b^2-a^2)}\left[b^2 M_b - a^2 M_a - (M_b - M_a)\frac{a^2 b^2}{r^2}\right] z. \end{aligned} \tag{5.14}$$

If the edge moments M_a and M_b are applied by a nonlinearly distributed radial stress along the hight h of the disk, the derived stress field approximately holds (in the spirit of Saint-Venant's principle) sufficiently away from the edges (e.g., for $a+h < r < b-h$). For a solid disk ($a = 0$), the constants $c_2 = 0$ and $c_1 = -(24/h^3)M_b/(1+\nu)$, and the stresses reduce to $\sigma_{rr} = \sigma_{\theta\theta} = 12M_b z/h^3$ (Timoshenko and Goodier, 1970).

The resulting stress expressions can be compared with the results from the classical circular plate theory (e.g., Timoshenko and Woinowsky-Krieger, 1987; Asaro and Lubarda, 2006). However, in contrast to the latter, the derived results in this section also apply to thick hollow cylinders loaded by distributed radial stress which varies linearly with z, giving rise to moments M_a and M_b.

The displacement components associated with (5.7) are

$$u_r = -\frac{1-\nu}{2\mu}\left(\frac{1}{2}c_1 r + \frac{c_2}{r}\right) z, \quad u_z = \frac{1-\nu}{2\mu}\left[\frac{1}{4}c_1(r^2 - b^2) + c_2 \ln(r/b)\right] + \frac{1}{2}\nu c_1 z^2, \tag{5.15}$$

where we have imposed the condition $u_z(r=b, z=0) = 0$.

Finally, we give the Boussinesq potentials corresponding to Love's potential (5.7). They can be expressed as

$$B = 2(1-\nu)[f(r) - g(z)]\,, \quad \beta = \frac{1}{2}\,(1-2\nu)\left[f(r) - \frac{1}{3}\,g(z)\right]z\,, \tag{5.16}$$

i.e., after using (5.5),

$$B = 2(1-\nu)\left[\frac{1}{4}\,c_1(r^2 - 2z^2) + c_2 \ln r\right], \;\; \beta = \frac{1}{2}\,(1-2\nu)\left(\frac{1}{4}\,c_1 r^2 + c_2 \ln r - \frac{1}{6}\,c_1 z^2\right)z\,. \tag{5.17}$$

6 Conclusions

We have presented in this article an analysis of classical Lamé problems of a symmetrically loaded or constrained hollow cylinder and a pressurized spherical container by using Love's and Boussinesq's potentials of the three-dimensional axisymmetric theory of elasticity. The presented analysis complements the original analysis based on the direct integration of the Navier equations of equilibrium, and the analysis based on the Airy stress function of the two-dimensional theory of elasticity. The simplest solution is based on the direct integration of the Navier equations, but the solutions based on the Airy stress function, and on Love's and Boussinesq's potentials are each conceptually and methodologically appealing on their own. An advantage of the solution based on the use of Love's and Boussinesq's potentials is that both the displacement and the stress components can be expressed in terms of these potentials explicitly and in closed form. The familiarity with the structure of Love's and Boussinesq's potentials for considered Lamé problems is also instructive for the analysis of more involved three-dimensional axisymmetric problems of the theory of elasticity, in which the structure of adopted potential functions is often constructed by inspection, through a semi-inverse or trial-and-error procedure. We also derived the stress and displacement fields in a hollow disk subjected to distributed bending moments along its circular boundaries.

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Rešavanje Lamé-ovih problema korišćenjem Love-ove potencijalne funkcije i Boussinesq-ovih potencijala

S a ž e t a k

U radu su izvedena rješenja Lamé-ovih problema osnosimetrično opterećenog šupljeg cilindra u uslovima ravne deformacije ili ravnog napona i sfernog suda pod pritiskom korišćenjem Love-ove potencijalne funkcije i Boussinesq-ovih potencijala trodimenzionalne osnosimetrične teorije elastičnosti. Ova rješenja predstavljaju dopunu klasičnih rješenja izvedenih direktnom integracijom Navier-ovih jednačina ravonoteže, ili korišćenjem Airy-eve naponske funkcije dvodimenzionalne teorije elastičnosti. Prednost Love-ove potencijalne funkcije i Boussinesq-ovih potencijala je da se i komponentalna pomjeranja i komponentalni naponi mogu eksplicitno izraziti korišćenjem ovih potencijala. Raspodjele napona i pomjeranja u šupljem disku opterećenom ravnomjerno raspoređenim ivičnim momentima savijanja su takođe određeni.

Ključne riječi: Airy-eva naponska funkcija; Boussinesq-ovi potencijali; kružna ploča; Lamé-ov problem; Love-ov potencijal; ravna deformacija; sferni rezervoar