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Time averaged optimization of cyclic processes with discount

Abstract

Here we prove the existence of solution in time averaged optimization of cyclic processes with discount and find the respective necessary optimality condition. We show that optimal strategy could be selected piecewise continuous if a differentiable profit density has a finite number of critical points. In such a case the optimal motion uses only maximum and minimum velocities as in Arnold's case without any discount.

Key words: optimal control, time averaged optimization, cyclic processes

1. Introduction

A control system on the cycle is defined by a velocity v smoothly depending on an angle x on the circle and a control parameter. We assume that this parameter belongs to a smooth closed manifold or a disjoint union of such ones with at least two different points and that all admissible velocities are positive, that is $v > 0$.

An *admissible motion* of the control system is an absolutely continuous map $x : t \mapsto x(t)$ from a time interval to the system phase space such that at each moment of differentiability of the map the motion velocity belongs to the convex hull of the admissible velocities of the system.

A *cyclic motion* or just a *cycle* with a period $T, T > 0$, is defined by a periodic admissible motion $x, x(t + T) \equiv x(t)$. In a presence of a continuous *profit density* f on the circle an optimization of periodic motion could lead to the problem of the selection of cyclic process with the maximum time averaged profit:

$$\frac{1}{T}\int_0^T f(x(t))dt \to \max .$$

V.I.Arnold shows that in a generic case the optimal strategy exists and is rather simple. Namely, it uses the maximum and minimum velocities when the profit density is less or greater, respectively, than a certain constant [1], [2], [3]. Here we prove an analogous result for the case with a positive discount σ, $\sigma > 0$:

$$\frac{1}{T}\int_0^T e^{-\sigma t} f(x(t))dt \to \max . \qquad (1)$$

2. Existence of optimal strategy

Here we prove the existence theorem in time averaged optimization of cyclic processes with discount.

2.1. Problem reformulation

Following V.I.Arnold [1] for an admissible motion x we introduce density $\rho, \rho(x(t)) = 1/\dot{x}(t)$. The density is well defined at any point of differentiability of the motion. So almost everywhere we have $dx(t) = \dot{x}(t)dt$ or, taking into account the positiveness of admissible velocities, $dt = \rho(x(t))dx(t)$.

Thus our extremal problem (1) could be rewritten in the form

$$A_\rho(f) := \int\limits_0^{2\pi} e^{-\sigma \int\limits_0^x \rho(z)dz} f(x)\rho(x)dx / \int\limits_0^{2\pi} \rho(x)dx \to \max$$

or

$$A_\rho(f) := (-\sigma)^{-1} \int\limits_0^{2\pi} f(x)d(e^{-\sigma\phi(x)}) / \int\limits_0^{2\pi} d(\phi(x)) \to \max . \qquad (2)$$

where $\phi(x) = \int\limits_0^x \rho(z)dz$. In such a formulation we need to find a measurable density ρ on the circle which satisfies the constraint

$$r_1 \le \rho \le r_2 \qquad (3)$$

and such that the respective function ϕ provides the maximum of averaged profit (2). Here r_1 and r_2 are positive functions being inverse values of maximum and minimum admissible velocity, and 0 and 2π are start and end points of the cycle, respectively.

2.2. Existence theorem

Left hand side of (2) could be considered as linear functional on the space of continuous profit densities.

Preposition 2.1. *For a given continuous density f and constraint positive functions r_1, r_2 the value of functional A_ρ for any measurable ρ satisfying constraint (3) is bounded. More exactly*

$$|A_\rho(f)| \le m_2 M/m_1$$

where $m_1 = \min\{r_1(x), 0 \le x \le 2\pi\}$, $m_2 = \max\{r_2(x), 0 \le x \le 2\pi\}$, $M = \max\{|f(x)|, 0 \le x \le 2\pi\}$.

Proof. Really for $|A_\rho(f)|$ we have the estimation

$$|A_\rho(f)| = \frac{|\int\limits_0^{2\pi} e^{-\sigma \int\limits_0^x \rho(z)dz} f(x)\rho(x)dx|}{|\int\limits_0^{2\pi} \rho(x)dx|} \le \frac{|\int\limits_0^{2\pi} M m_2 dx|}{|\int\limits_0^{2\pi} m_1 dx|} = \frac{m_2 M}{m_1}.$$

Theorem 2.2. *For a continuous profit density f and continuous positive constraint functions r_1, r_2 there exists a measurable density ρ_{max} which satisfies constraint (3) and provides exact upper bound of values $A_\rho(f)$ over all measurable functions ρ complying with this constraint.*

Proof. Let for a sequence of measurable densities ρ_n the value $A_{\rho_n}(f)$ tend to this upper bound when $n \to \infty$. The respective sequence ϕ_n of functions $\int\limits_0^x \rho_n(z)dz$, $x \in [0, 2\pi]$, satisfies the condition

$$m_1(y - x) \le \phi_n(y) - \phi_n(x) \le m_2(y - x) \tag{4}$$

for any $x, y \in [0, 2\pi]$, $x \le y$, due to ρ_n satisfies constraint (3). In particular, all ϕ_n are Lipschitz functions with the same constant m_2, on the interval $[0, 2\pi]$ the set of these functions is bounded and has equicontinuity property.

Consequently, due to Arcela-Askoli theorem there is subsequence of these functions which converges uniformly on the interval $[0, 2\pi]$ to a function ϕ_∞. Taking the respective limit in inequality (4) we observe that the function ϕ_∞ has to satisfy this inequality.

Therefore this function is absolutely continuous. Its derivative exists almost everywhere and has to satisfy constraint (3) at any point of its existence. Hence there exists the needed measurable function ρ_{max} which coincides with the derivative of ϕ_∞ at any point of its differentiability.

The theorem is proved.

3. Structure of optimal solution

Here we calculate a necessary optimality condition. Using it we show that a optimal density ρ_{max} is bang-bang and find the form of the respective switching function.

3.1. Necessary optimality condition

Theorem 3.1. *If for a continuous profit density f and continuous positive constraint functions r_1, r_2 the maximum A of functional in (2) is provided by a density ρ satisfying constraint (3) then at any point x, where ρ is derivative of its integral, the value*

$$e^{-\sigma \int\limits_0^x \rho(z)dz} f(x) - \sigma \int\limits_x^{2\pi} e^{-\sigma \int\limits_0^y \rho(z)dz} f(y)\rho(y)dy - A \tag{5}$$

is either non-positive or non-negative, or else zero if the value $\rho(x)$ is equal to either $r_1(x)$ or $r_2(x)$, or else belongs to $(r_1(x), r_2(x))$, respectively.

Proof. The statement of the theorem follows from direct calculation of the first variation of functional value $A_\rho(f)$ under admissible appropriate variation of the density ρ. Consider a point x at which the function ρ equals to derivative of its integral. Without loss of generality we assume that this point belongs to the interval $(0, 2\pi)$. For the ends of the interval the proof is practically the same.

Take a small positive ν such that the interval $[x, x+\nu]$ is subset of $(0, 2\pi)$ and consider a new density $\tilde{\rho}$ such that the difference $\tilde{\rho} - \rho$ is a small constant value h on this interval and zero outside it. For the densities ρ and $\tilde{\rho}$ the respective periods of motion along the circle are

$$T = \int_0^{2\pi} \rho(x)dx \quad \text{and} \quad \tilde{T} = T + h\nu$$

as it is easy to see. Calculating now the parts of difference $A_{\tilde{\rho}}(f) - A_\rho(f)$ which are related to intervals $[0, x]$, $[x, x+\nu]$, and $[x+\nu, 2\pi]$ we get respectively

$$\frac{\int\limits_0^x e^{-\sigma \int\limits_0^y \rho(z)dz} f(y)\rho(y)dy}{T + h\nu} - \frac{\int\limits_0^x e^{-\sigma \int\limits_0^y \rho(z)dz} f(y)\rho(y)dy}{T} =$$

$$= -\frac{h\nu}{T} \times \frac{\int\limits_0^x e^{-\sigma \int\limits_0^y \rho(z)dz} f(y)\rho(y)dx}{T} + \dots, \qquad (6)$$

$$\frac{\int\limits_x^{x+\nu} e^{-\sigma \int\limits_0^y (\rho(z)+h)dz} f(y)(\rho(y)+h)dy}{T+h\nu} - \frac{\int\limits_x^{x+\nu} e^{-\sigma \int\limits_0^x \rho(z)dz} f(x)\rho(x)dx}{T} =$$

$$= \frac{h\nu}{T} \times e^{-\sigma \int\limits_0^x \rho(z)dz} f(x) + \dots, \qquad (7)$$

$$\frac{\int\limits_{x+\nu}^{2\pi} e^{-\sigma[\int\limits_0^y \rho(z)dz + h\nu]} f(y)\rho(y)dy}{T+h\nu} - \frac{\int\limits_{x+\nu}^{2\pi} e^{-\sigma \int\limits_0^y \rho(z)dz} f(y)\rho(y)dy}{T} =$$

$$= -\frac{h\nu}{T} \times (\sigma + \frac{1}{T}) \int\limits_x^{2\pi} e^{-\sigma \int\limits_0^y \rho(z)dz} f(y)\rho(y)dy + \dots, \qquad (8)$$

where dots stay for the higher degree terms with respect to h and ν. The sum of the righthand sides of (6), (7) and (8) gives the whole difference $A_{\tilde{\rho}}(f) - A_\rho(f)$ in the form

$$\frac{h\nu}{T} \times \left[e^{-\sigma \int\limits_0^x \rho(z)dz} f(x) - \sigma \int\limits_x^{2\pi} e^{-\sigma \int\limits_0^y \rho(z)dz} f(y)\rho(y)dy - A \right] + \dots \qquad (9)$$

with $A = A_\rho(f)$.

Thus for a small h and $\nu > 0$ the sign of the value of this difference is defined by the sign of product h with the expression in square brackets because both the period T and ν are positive. For an optimal density ρ this value should be non-positive at any point where this density equals to derivative of its integral.

Consequently the expression in square brackets has to be non-positive, non-negative or zero if the value $\rho(x)$ is equal to either $r_1(x)$ or $r_2(x)$, or else belongs to $(r_1(x), r_2(x))$, respectively, because in these three cases we could take a small value of h as non-negative, non-positive and both non-negative and non-positive, respectively.

Thus the statement of Theorem 3.1. is true.

3.2. Analysis of switching function

Expression (5) defines a function $S, S = S(x)$. In some sense this function plays the role of switching function. It could be rewritten as

$$e^{-\sigma\int\limits_0^x \rho(z)dz} f(x) + \sigma \int\limits_0^x e^{-\sigma\int\limits_0^y \rho(z)dz} f(y)\rho(y)dy - \sigma P - A, \tag{10}$$

where $P = \int\limits_0^{2\pi} e^{-\sigma\int\limits_0^y \rho(z)dz} f(y)\rho(y)dy$ is the profit along the cycle. Note that for $\sigma = 0$ this function takes well known form [1], [2], [3].

For differentiable profit density the switching function takes the form

$$f(0) + \int_0^x e^{-\sigma\int_0^y \rho(z)dz} f'(y)dy - \sigma P - A \tag{11}$$

after an integration in parts in (10). Here T is the period of the cycle.

Form (11) of function S leads to

Preposition 3.2. *For a differentiable profit density f, continuous positive constraint functions r_1, r_2 and a measurable density ρ satisfying (3) the function S is differentiable and has the same critical points with the function f.*

Proof. Taking the switching function S in the form (11) we get

$$S'(x) = e^{-\sigma\int_0^x \rho(x)dz} f'(x)$$

because in (11) the integrant is continuous function. But the exponent never vanishes. Consequently the derivatives of the functions f and S have the same zeros.

Theorem 3.3. *For a differentiable profit density f with a finite number k of critical points an optimal density ρ could be taken as piecewise continuous function. More exactly, in such a case the optimal density takes either value r_1 or r_2 inside any interval between consequent points from the zeros of the switching function and the ends $0, 2\pi$.*

Proof. When the differentiable profit density has a finite number k of zeros of its derivative on the interval $[0, 2\pi]$ then the switching function S does that too. Consequently the switching function could vanish at least at k points inside this interval. So in such a case an optimal density ρ takes values either ρ_1 or ρ_2 inside any interval between consequent points from the zeros of the function S and the ends $0, 2\pi$.

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