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ON A CONTINUOUS METHOD WITH CHANGEABLE METRIC FOR SOLVING VARIATIONAL INEQUALITIES

A b s t r a c t

We study one continuous method with changeable metric for solving variational inequalities and establish sufficient conditions for the convergence of the proposed method.

Key words. Continuous methods, changeable metric

O JEDNOM NEPREKIDNOM METODU SA PROMJENLJIVOM METRIKOM ZA RJEŠAVANJE VARIJACIONIH NEJEDNAKOSTI

I z v o d

U radu se izučava jedan neprekidni metod sa promjenljivom metrikom za rješavanje varijacionih nejednakosti i formulišu dovoljni uslovi za konvergenciju tog metoda.

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1. Introduction. Let X be a real normed linear space and K-convex closed cone in X. Let us suppose that the subset U of a real Hilbert space H is given by

$$U = \{x \in U_0 : -g(x) \in K\} \tag{1}$$

where $U_0 \subseteq H$ is a closed convex set and $g : H \to X$ is a differentiable map that satisfies condition of the convexity:

$$(\forall x, y \in H)(\forall \lambda \in [0,1])\ \lambda g(x) + (1-\lambda)g(y) - g(\lambda x + (1-\lambda)y)) \in K.$$

If we define the relation "$\leq$" on the space X by

$$x \leq y \text{ if and only if } y - x \in K,$$

then the set U can be described as

$$U = \{x \in H : g(x) \leq 0\}. \tag{1'}$$

We assume we have a mapping $F : H \to H$ and we seek a solution x_* of the following variational inequality:

$$\langle F(x_*), x - x_* \rangle \geq 0, for all x \in U. \tag{2}$$

In what follows, we will propose one continuous method for solving this problem and we will establish some sufficient condition of the convergence of the proposed method. Among other conditions, we will suppose that the operator $F : H \to H$ is continuous and that it satisfies the condition of the monotonicity on U_0 :

$$\langle F(x) - F(y), x - y \rangle \geq 0, \quad for all x, y \in U_0 \tag{3}$$

Further, we will assume that

$$U_* = \{x_* \in U : \langle F(x_*), x - x_* \rangle \geq 0, \forall x \in U\} \neq \emptyset. \tag{4}$$

and that the constraint-set U satisfies the Slater assumption:

$$\exists x_c \in U_0 : -g(x_c) \in int\, K. \tag{8}$$

2. Method. Let G be a mapping from H to $\mathcal{L}(\mathcal{H},\mathcal{H})$, such that

$$\forall x \in H \; G(x) \text{ is self-adjoint and positive linear operator} \tag{5}$$

Then, for solving problem (1), (2), one can use method with changeable metric:

$$x'(t)+x(t)=\mathcal{P}_{U(x(t))}^{G(x(t))}[x(t)-\gamma(t)G^{-1}(x(t))F(x(t))], t>0, x(0)=x_0 \tag{6}$$

$$U(x(t))=\{y :\in U_0 : g(x(t))+g'(x(t))(y-x(t))\leq 0\} \tag{7}$$

where $\gamma(t)\geq 0, t\geq 0$; x_0 is a given initial point; $G^{-1}(x(t))$ is the inverse operator of the operator $G(x(t))$; $P_{U(x)}^{G(x)}(z)$ is the projection of the point z on the set $U(x)$ in the norm $||y||_{G(x)}=\langle G(x)y,y\rangle^{\frac{1}{2}}$.

Let us remark that for every $t>0, U\subseteq U(x(t))\subseteq U_0$. This implies that the closed convex set $U(x(t))\subseteq H$ is nonempty and the projection in (6) is well defined. According to the properties of the projecting operator ([1], p. 183),the relation (6) is equivalent to the following variational inequality

$$\langle G(x(t))x'(t)+\gamma(t)F(x(t)), x'(t)+x(t)-z\rangle\geq 0, z\in U(x(t)), t>0. \tag{9}$$

Note that for $F(x)=f'(x), H=\mathbf{R}^n,\; X=\mathbf{R}^m,$ the method (6),(7) turns into continuous linearization method with changeable metric [2]. Some classes of continuous methods of minimization, close to proposed, were considered in [3] and [4].

3. Convergence of the method. Under these assumptions, the Lagrangian functions for the problems (1),(2) and (9) have saddle points (x_*,λ^*) and $(x'(t)+x(t),\nu(t))$ ([5], p. 74), such that

$$\lambda^*\geq 0, \lambda^*\neq 0, x_*\in U_*; \tag{10}$$

$$(\forall y\in U_0)\;\langle F(x_*), y-x_*\rangle+\langle\lambda^*, g'(x_*)(x_*-y)\rangle\leq 0; \tag{11}$$

$$g(x_*)\leq 0,\; \langle\lambda^*, g(x_*)\rangle=0, \tag{12}$$

$$\nu(t)\geq 0, \nu(t)\neq 0, t\geq 0 \tag{13}$$

$$(\forall z\in U_0)\;(\forall t>0)\;\langle G(x(t))x'(t)+\gamma(t)F(x(t)), x'(t)+x(t)-z\rangle+$$
$$\langle\nu(t), g'(x(t))(x'(t)+x(t)-z\rangle\leq 0, \tag{14}$$

$$(\forall t > 0)\ \langle \nu(t), g(x(t)) + g'(x(t))x'(t)\rangle = 0, \tag{15}$$

$$(\forall t > 0)\ g(x(t)) + g'(x(t))x'(t) \leq 0. \tag{16}$$

Theorem 1. *Suppose that all the following conditions be fulfilled:*

(1) U_0 is closed convex set in Hilbert space H; g is a differentiable and convex mapping from Hilbert space H to a normed linear space X, satisfying (8); $F : H \to H$ is a continuous mapping satisfying the condition of monotonicity (3) and Lipschitz condition

$$\max\{\|F(x) - F(y)\|; \|g'(x) - g'(y)\| \leq L\|x - y\|\}, L > 0, x, y \in H. \tag{17}$$

such that (4) is valid.

(2) the mapping G satisfies (5) and for every $x \in H$ there is a strongly convex function $\psi \in C^2(H)$ satisfying

$$\psi"(x) = G(x), m\|\xi\|^2 \leq \langle G(x)\xi, \xi\rangle \geq M\|\xi\|^2, m > 0, x, \xi \in H \tag{18}$$

(3) $\gamma \in C[0, +\infty)$ is a continuous function satisfying

$$0 < \gamma_0 \leq \gamma(t) \leq \gamma_1, t \geq 0, 3m - \gamma_1 L \cdot (1 + 2\|\lambda^*\|) > 0. \tag{19}$$

Then the set $\{x(t) : t \in [0, +\infty)\}$ is bounded and

$$\liminf_{t\infty} \|x'(t)\| = 0.$$

In the case $H = \mathbf{R}^m, X = \mathbf{R}^n$ there exists a point x_∞ such that

$$\lim_{t\to\infty} \{\|x(t) - x_\infty\| + \|x'(t)\|\} = 0.$$

Proof. First, note that a convex differentiable function f satisfies the following inequality [1, p. 160]

$$f(x) + \langle f'(x), y - x\rangle \leq f(y), x, y \in H \tag{20}$$

Setting $z = x_* \in U_* \subseteq U_0$ in (14), $y = x'(t) + x(t) \in U(x(t)) \subseteq U_0$ in (11) and multiplying (11) by $\gamma(t) > 0$, in the sum of the obtained inequalities we will have

$$\langle G(x(t))x'(t), x'(t) + x(t) - x_*\rangle + \langle \nu(t), g'(x(t))(x'(t) + x(t) - x_*)\rangle \leq$$

$$\gamma(t)\langle F(x(t)) - F(x_*), x_* - x'(t) - x(t)\rangle +$$
$$\langle \lambda^*, g'(x_*)(x'(t) + x(t) - x_*)\rangle, t > 0, x_* \in U_*. \tag{21}$$

Now, we are going to estimate some of the terms in (21).

First, we will show that

$$\langle \nu(t), g'(x(t))(x'(t) + x(t) - x_*)\rangle \geq 0, t > 0, x_* \in U_*. \tag{22}$$

Combining (13), (15) and (20), we have

$$\langle \nu(t), g(x(t))(x'(t) + x(t) - x_*)\rangle = \langle \nu(t), g'(x(t))(x'(t)) + g(x(t))\rangle -$$
$$\langle \nu(t), g(x(t)) + g'(x(t))(x_* - x(t))\rangle \geq -\langle \nu(t), g(x_*)\rangle \geq 0,$$

for all $t > 0, x_* \in U_*$. Thus, (22) is hold.

Second, we will prove the inequality

$$\langle \lambda^*, g'(x_*)(x'(t) + x(t) - x_*)\rangle \leq \frac{L}{2}\|\lambda^*\| \cdot \|x'(t)\|^2, \tag{23}$$

for $t > 0, x_* \in U_*$.

Combining (12) and (20) we obtain

$$\langle \lambda^*, g'(x_*)(x'(t) + x(t) - x_*)\rangle \leq$$
$$\langle \lambda^*, g(x'(t) + x(t)) - g(x_*)\rangle = \langle \lambda^*, g(x'(t) + x(t))\rangle, t > 0, x_* \in U_*. \tag{24}$$

For every differential function $f(\cdot)$, whose gradient satisfies Lipschitz condition, the following estimate has a place ([1], p. 87)

$$\langle \lambda^*, f(y) - f(z) - \langle f'(z), y - z\rangle \leq \frac{L}{2}\|\lambda^*\| \cdot \|y - z\|^2, y, z \in H. \tag{25}$$

If we combine this with (10) and (16), we get

$$\langle \lambda^*, g(x'(t) + x(t))\rangle = \langle \lambda^*, g(x'(t) + x(t)) - g(x(t)) - g'(x(t))x'(t)\rangle +$$
$$\langle \lambda^*, g(x(t)) + g'(x(t))x'(t)\rangle \leq \|\lambda^*\|\frac{L}{2}\|x'(t)\|^2, t > 0.$$

Now, (23) follows from this inequality and (24).

Since the continuous mapping F satisfies (3) and (17), the following estimate is true (see [1], p. 170, Theorem 16)

$$\langle F(x) - F(y, y - z)\rangle \leq \frac{L}{4}\|x - z\|^2, x, y, z \in H. \tag{26}$$

Putting the estimation (22), (23) and (26) in the inequality (21), we obtain

$$\langle G(x(t))x'(t), x'(t)\rangle + \langle G(x(t))x'(t), x(t) - x_*\rangle \leq c\gamma(t)\|x'(t)\|^2, \quad (27)$$

for $t > 0, x_* \in U_*$, with $c = \frac{L}{4}(1 + 2\|\lambda^*\|)$.

So, taking into account (19), we obtain

$$(m - \gamma_1 c)\|x'(t)\|^2 + \langle G(x(t))x'(t), x(t) - x_*\rangle \leq 0, t > 0, x_* \in U^*. \quad (28)$$

On the other hand, the first relation in (18) gives

$$\langle G(x(t))x'(t), x(t)-x_*\rangle = \frac{d}{dt}(\psi(x_*)-\psi(x(t))+\langle\psi'(x(t)), x(t)-x_*\rangle) \leq 0.$$

Therefore, (28) becomes

$$(m - \gamma_1 c)\|x'(t)\|^2 + \frac{d}{dt}(\psi(x_*) - \psi(x(t)) + \langle\psi'(x(t)), x(t) - x_*\rangle) \leq 0.$$

for $t > 0, x_* \in U_*$. Integrating this inequality on segment $[\xi, t], t > \xi > 0$, we obtain

$$(m-\gamma_1 c)\int_\xi^t \|x'(s)\|^2 ds + \psi(x_*) - \psi(x(t)) + \langle\psi'(x(t)), x(t)-x_*\rangle \leq K(\xi, x_*), \quad (29)$$

where

$$K(\xi, x_*) = \psi(x_*) - \psi(x(\xi)) - \langle\psi'(x(\xi)), x(\xi) - x_*\rangle, \quad (30)$$

for all $t > \xi > 0, x_* \in U_*$. Since $\psi(\cdot)$ is a strongly convex function, it follows that

$$\psi(x_*) - \psi(x(t)) + \langle\psi'(x(t)), x(t) - x_*\rangle \geq \frac{m}{2}\|x(t) - x_*\|^2 t > 0, x_* \in U_*.$$

This, together with (29), leads us to

$$\int_0^{+\infty} \|x'(s)\|^2 ds < +\infty \Rightarrow \liminf_{t\to\infty} \|x'(t)\| = 0,$$

$$\limsup_{t\to+\infty} \|x(t) - x_*\|^2 \leq \frac{2}{m} \cdot K(\xi, x_*), \xi > 0, x_* \in U_*. \quad (31)$$

So, we proved the first part of the Theorem.

Now, let us consider the case $H = \mathbf{R}^n, X = \mathbf{R}^m$, $g = (g_1, \ldots, g_n)$, $\lambda = (\lambda_1, \ldots, \lambda_m), \nu(t) = (\nu_1(t), \ldots, \nu_m(t))$. Since the set $\{x(t) : t > 0\}$ is bounded, there are a sequence $(t_j, j \in N) \subseteq [0, +\infty)$, a real numbers $\gamma_\infty > 0, \nu_{i\infty} \geq 0, i = 1, \ldots, m$, and a point $x_\infty \in \mathbf{R}^n$ such that

$$\liminf_{t\to\infty} \|x'(t)\| = \lim_{j\to\infty} \|x'(t_j)\| = 0, \liminf_{t\to\infty} \gamma(t_j) = \gamma_\infty > 0,$$

$$\lim_{j\to\infty} \|x(t_j) - x_\infty\| = 0, \lim_{j\to\infty} \frac{\nu_i(t_j)}{\gamma(t_j)} = \frac{\nu_{i\infty}}{\gamma_\infty} = \nu_i^* \geq 0, i = 1, \ldots, m. \quad (32)$$

Setting $t = t_j, j \to \infty$, in (14)-(16), we get

$$\gamma_\infty \langle F(x_\infty) + \sum_{i=1}^m \nu_i^\infty g_i'(x_\infty), x_\infty - z\rangle \leq 0, \forall z \in U_0.$$

These means [1, p. 219] that $x_\infty \in U_*$. Putting $\xi = t_j$, and $x_* = x_\infty \in U_*$ in (30) when $j \to \infty$, we obtain

$$\lim_{j\to\infty} K(t_j, x_\infty) = 0.$$

Combining this with (31) (for $\xi = t_j, x_* = x_\infty, j \to \infty$,) we conclude that

$$\lim_{t\to\infty} \|x(t) - x_\infty\| = 0.$$

Finally, we are going to prove that $\lim_{t\to\infty} \|x'(t)\| = 0$. Using (28) we have

$$(m - \gamma_1 c)\|x'(t)\|^2 \leq \langle G(x(t))x'(t), x_* - x(t)\rangle \leq$$

$$2\left(\frac{\sqrt{m}}{2}\|x'(t)\|\right)\left(\frac{1}{\sqrt{m}}\|x(t) - x_*\|\|G(x(t))\|\right) \leq$$

$$\frac{m}{4}\|x'(t)\|^2 + \frac{1}{m}\|x(t) - x_*\|^2\|G(x(t))\|^2, t > 0, x_* \in U_*.$$

Setting here $x_* = x_\infty, t \to \infty$, we get

$$\frac{1}{4}(3m - 4\gamma_1 C)\limsup_{t\to\infty} |x'(t)\|^2 \leq 0.$$

This, (19) and (31) provide that $\lim_{t\to\infty} \|x'(t)\| = 0$. This completes the proof of the Theorem.

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