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SLIDING AND BONDED CIRCULAR INCLUSIONS IN CONCENTRIC CYLINDERS

Abstract

Exact elasticity solutions for circular inclusions with sliding and bonded interface within a concentric circular cylinder are obtained, associated with uniform shear eigenstrain of the inclusion or remote shear loading of the cylinder. Solutions for the bonded and sliding inhomogeneities are also given. Elastic strain energies are calculated in each case for an infinitely extended matrix surrounding the inclusion or the inhomogeneity. The results are important for evaluation of the effects that interface conditions have on the average elastic properties of fiber composites and related problems.

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KRUŽNE INKLUZIJE SA KLIZNIM I ČVRSTIM SPOJEM U KONCENTRIČNIM CILINDRIMA

Izvod

Naponska i deformaciona stanja unutar i izvan kružnih inkluzija umetnutih u koncentrični cilindar su odredjena za slučaj zadate smičuće sopstvene-deformacije u inkluziji ili zadatog smičućeg opterećenja na spoljnjoj konturi cilindra. Granični uslovi izmedju inkluzije i cilindra odgovaraju kliznom ili čvrstom spoju, zavisno od toga da li je dopušten diskontinuitet u tangencijalnoj komponenti pomjeranja na granici izmedju inkluzije i cilindra. Elastične deformacione energije su sračunate i uticaj prirode spoja na njihove vrijednosti je analiziran. Dobijeni rezultati su od značaja za odredjivanje srednjih vrijednosti elastičnih svojstava kompozitnih materijala ojačanih vlaknima u zavisnosti od vrste spoja na granici izmedju vlakana i matrice materijala.

INTRODUCTION

Since the classic paper by Eshelby (1957) on the elastic field of an ellipsoidal inclusion due to uniform eigenstrain, or an inhomogeneity under remote loading, the study of elastic inclusions and inhomogeneities has kept a continuing attention of the mechanics researchers over a period of last four decades. The books by Mura (1987) and Nemat–Nasser (1993) illustrate well the amount of research done and its significance in modeling of various engineering problems in solid mechanics and materials science. Most solutions given in the literature are for inclusions and inhomogeneities within infinite media, for which the nature of remote boundary conditions simplifies the analysis and often allows an exact, closed form solution. It is the purpose of this paper to study inclusions and inhomogeneities in finite media, and to determine the effects of the remote boundary on the stress field and strain energy. A circular inclusion which has undergone a uniform eigenstrain transformation of shear type is placed in a concentric circular cylinder, unloaded over its external boundary. An inclusion is also considered in a cylinder loaded over its external surface by tractions associated with the uniform state of shear stress. The same is done for an inhomogeneity. Plane strain conditions are assumed to prevail. Two types of interface conditions are studied in detail, perfect bonding and frictionless or slipping interface. Exact solutions are found in all considered cases by employing appropriate Papkovich–Neuber potential functions. Elastic strain energies in the inclusion or inhomogeneity, and in the surrounding matrix, are determined. Their dependence on the interface conditions is examined. Obtained results are useful for evaluation of the average elastic properties of fiber composites, modeling of the grain boundary sliding in polycrystalline materials, and related problems (Ghahremani 1980, Jasiuk *et al.* 1987, Furuhashi *et al.* 1992, Lubarda 1997, Lubarda and Markenscoff 1998,1999). Extension of the analysis to include an arbitrary uniform eigenstrain of the inclusion, or an arbitrary uniform loading over external surface of the cylinder, can be done much along the lines presented in this paper.

1. CIRCULAR INCLUSION WITH SHEAR EIGENSTRAIN

Consider a circular inclusion of radius a within a concentric circular cylinder of radius R. Let the inclusion be given a stress-free transformation strain (eigenstrain) of shear type and amount γ, relative to the set of specified rectangular axes x and y. Upon insertion of the transformed inclusion back into the cylinder, both are in the state of stress and strain. The Papkovich–Neuber potentials for the displacements in the inclusion can be taken as:

$$\Phi_0 = (b_0 + a_1)r^2 \sin 2\theta, \ \Phi_1 = a_2 r^3 \sin 3\theta, \ \Phi_2 = -a_2 r^3 \cos 3\theta, \quad (1.1)$$

where r and θ denote the polar coordinates. The displacement components are derived from:

$$u_x = \frac{\partial}{\partial x}(\Phi_0 + x\Phi_1 + y\Phi_2) - 4(1-\nu)\Phi_1 \mid +\gamma y, \tag{1.2}$$

$$u_y = \frac{\partial}{\partial y}(\Phi_0 + x\Phi_1 + y\Phi_2) - 4(1-\nu)\Phi_2 \mid +\gamma x. \tag{1.3}$$

The terms γx and γy, appearing to the right of the vertical ($|$) line, correspond to stress-free shear eigenstrain, and should not be taken into account when calculating the stresses. The plane strain configuration is assumed, and ν is the Poisson ratio of an isotropic material of the inclusion and the cylinder.

The Papkovich–Neuber potentials for displacements in the cylinder are:

$$\begin{aligned} \Phi_0 &= b_0 r^2 \sin 2\theta + b_1\theta + b_2 r^{-2}\sin 2\theta, \\ \Phi_1 &= b_3 r^{-1}\sin\theta + b_4 r^3 \sin 3\theta, \\ \Phi_2 &= -b_4 r^3 \cos 3\theta. \end{aligned} \tag{1.4}$$

The corresponding displacement components are obtained from (1.2) and (1.3), excluding γx and γy terms on their right-hand sides. The following displacement and stress components result in polar coordinates. The displacements in the inclusion are:

$$u_r^I = \left[2\left(b_0 + a_1 + 2\nu a_2 r^2\right) + \gamma\right] r \sin 2\theta, \tag{1.5}$$

$$u_\theta^I = \left\{2\left[b_0 + a_1 + (3-2\nu)\, a_2 r^2\right] + \gamma\right\} r \cos 2\theta, \tag{1.6}$$

and the stresses:

$$\sigma_r^I = 4\mu\,(b_0 + a_1)\sin 2\theta, \tag{1.7}$$

$$\sigma_\theta^I = -4\mu\left(b_0 + a_1 + 6a_2 r^2\right)\sin 2\theta, \tag{1.8}$$

$$\sigma_{r\theta}^I = 4\mu\left(b_0 + a_1 + 3a_2 r^2\right)\cos 2\theta. \tag{1.9}$$

The displacement components in the cylinder are similarly:

$$u_r^C = 2\left[b_0 - b_2 r^{-4} - (1-\nu)\, b_3 r^{-2} + 2\nu b_4 r^2\right] r \sin 2\theta, \tag{1.10}$$

$$u_\theta^C = 2\left[b_0 + b_2 r^{-4} - \frac{1}{2}(1-2\nu)\, b_3 r^{-2} + (3-2\nu)\, b_4 r^2\right] r\cos 2\theta, \tag{1.11}$$

with the corresponding stresses:

$$\sigma_r^C = 4\mu\left(b_0 + 3b_2 r^{-4} + b_3 r^{-2}\right)\sin 2\theta, \tag{1.12}$$

$$\sigma_\theta^C = -4\mu\left(b_0 + 3b_2 r^{-4} + 6b_4 r^2\right)\sin 2\theta, \tag{1.13}$$

$$\sigma_{r\theta}^C = 4\mu\left(b_0 - 3b_2 r^{-4} - \frac{1}{2}b_3 r^{-2} + 3b_4 r^2\right)\cos 2\theta, \tag{1.14}$$

provided that $b_1 = -2(1-\nu)b_3$. The shear modulus is denoted by μ.

If the interface between the inclusion and the cylinder is bonded, the interface conditions are the continuity of the normal and shear traction, and the normal and tangential displacement. The boundary conditions at the outer surface of the cylinder are the vanishing of the normal and shear traction there. Thus, at $r = a$:

$$\sigma_r^I = \sigma_r^C, \quad \sigma_{r\theta}^I = \sigma_{r\theta}^C, \quad u_r^I = u_r^C \quad u_\theta^I = u_\theta^C, \tag{1.15}$$

and at $r = R$:

$$\sigma_r^C = 0, \quad \sigma_{r\theta}^C = 0. \tag{1.16}$$

The superscript I designates the inclusion, and C the surrounding cylinder or the matrix material. Upon calculations, the following expressions for the constants a's and b's are obtained:

$$a_1 = -k\gamma, \quad a_2 = -2k\gamma\frac{a^2}{R^4}\left(1 - \frac{a^2}{R^2}\right), \tag{1.17}$$

$$b_0 = k\gamma\frac{a^2}{R^2}\left(4 - 3\frac{a^2}{R^2}\right), \; b_1 = \gamma a^2, \; b_2 = k\gamma a^4, \; b_3 = -4k\gamma a^2, \tag{1.18}$$

and $b_4 = a_2$, where $k = 1/8(1-\nu)$. A discontinuity in the hoop stress across the interface of the bonded inclusion is equal to $\Delta\sigma_\theta =$

$-16\mu k\gamma \sin 2\theta$. This is independent of the ratio a/R, and thus equal to the discontinuity in the hoop stress of the Eshelby bonded inclusion in an infinite matrix.

If the interface between the inclusion and the cylinder is frictionless, perfectly slipping interface, the inclusion is commonly referred to as the sliding inclusion (Jasiuk *et al.* 1987, Furuhashi *et al.* 1992, Lubarda and Markenscoff 1998). The interface conditions are the vanishing of the shear traction between the inclusion and the cylinder, and the continuity of the normal traction and normal displacement at the interface. Thus, at $r = a$:

$$\sigma_{r\theta}^I = 0, \quad \sigma_{r\theta}^C = 0, \quad \sigma_r^I = \sigma_r^C, \quad u_r^I = u_r^C. \tag{1.19}$$

The boundary conditions at the outer surface of the cylinder are the same as in the case of the bonded inclusion, i.e. given by Eq. (1.16). Upon a somewhat lengthy but straightforward calculation, the following expressions for the constants a's and b's are found:

$$a_1 = -\frac{3k\gamma}{2\varrho}\left(1 - \frac{a^2}{R^2} + 4\frac{a^4}{R^4}\right), \quad a_2 = \frac{k\gamma a^{-2}}{2\varrho}\left(1 - \frac{a^2}{R^2}\right)^3, \tag{1.20}$$

$$b_0 = \frac{3k\gamma}{2\varrho}\frac{a^2}{R^2}\left(2 + \frac{a^2}{R^2} + \frac{a^4}{R^4}\right), \quad b_2 = \frac{k\gamma a^4}{2\varrho}\left(1 + 3\frac{a^2}{R^2}\right), \tag{1.21}$$

$$\begin{aligned} b_3 &= -\frac{3k\gamma a^2}{\varrho}\left(1 + \frac{a^2}{R^2} + 2\frac{a^4}{R^4}\right), \\ b_4 &= -\frac{k\gamma R^{-2}}{2\varrho}\frac{a^2}{R^2}\left(3 + \frac{a^2}{R^2}\right), \end{aligned} \tag{1.22}$$

where $\varrho = 1 + 3a^4/R^4$. A discontinuity in the tangential displacement across the slipping interface is

$$\Delta u_\theta = u_\theta^C(a,\theta) - u_\theta^I(a,\theta) = -\frac{\gamma}{2\varrho}\left(1 - 3\frac{a^2}{R^2} + 6\frac{a^4}{R^4}\right) a \cos 2\theta. \tag{1.23}$$

A discontinuity in the hoop stress across the interface of the sliding inclusion is

$$\Delta\sigma_\theta = \sigma_\theta^C(a,\theta) - \sigma_\theta^I(a,\theta) = -\frac{48\mu k\gamma}{\varrho}\frac{a^2}{R^2}\left(1 - \frac{a^2}{R^2}\right)\sin 2\theta. \tag{1.24}$$

If $R \to \infty$, then $\Delta\sigma_\theta = 0$, which shows that there is no jump in the hoop stress across the interface of the sliding inclusion and an infinite matrix. This is in contrast to the Eshelby inclusion with bonded interface, where the hoop stress experiences a variable jump $\Delta\sigma_\theta = -16\mu k\gamma \sin 2\theta$. Also, the stress state at all points of the interface between the sliding inclusion and the surrounding infinite matrix is purely dilatational, in the sense that $\sigma_r = \sigma_\theta$ at the interface. The normal tractions at the interface of sliding and bonded inclusions in infinite matrix are related by $\sigma_r^S = (3/2)\sigma_r^B$, so that upon removal of the shear traction at the interface of the bonded inclusion, the normal traction there increases by the factor of 3/2, to preserve the continuity of the normal displacement across the interface.

2. SLIDING CIRCULAR INCLUSION UNDER REMOTE SHEAR LOADING

Consider a cylinder of radius R which contains a stress-free concentric circular inclusion of radius a. Both materials are the same, and the interface between the inclusion and the cylinder is frictionless. Assume that the loading at the outer surface of the cylinder corresponds to the state of uniform shear stress $\sigma_{xy} = \tau$, so that the boundary conditions at $r = R$ are:

$$\sigma_r^C = \tau \sin 2\theta, \quad \sigma_{r\theta}^C = \tau \cos 2\theta. \tag{2.1}$$

It is important to consider such a configuration in order to properly evaluate the relevant elastic strain energies in the problem, discussed in Section 4. An analogous but simpler problem, with a circular void in a finite cylinder under the same loading, was originally solved by Sih and Liebowitz (1967) to confirm the Griffith energy criterion for brittle fracture. The Papkovich–Neuber potentials can be taken in the general form given by Eqs. (1.1) and (1.4), with the corresponding displacement and stresses defined by Eqs. (1.5)–(1.14), omitting the terms proportional to γ. Alternatively, the complex functions and the Kolosov–Muskhelishvili stress combinations can be used. The constants a's and b's appearing in Eqs. (1.5)–(1.14) are found to be:

$$a_1 = \alpha\left[3 - 4\frac{a^2}{R^2} - 3\beta\left(1 - \frac{a^2}{R^2} + 4\frac{a^4}{R^4}\right)\right], \tag{2.2}$$

$$a_2 = -\alpha a^{-2}\left[2\left(1 - \frac{a^2}{R^2}\right) - \beta\left(1 - \frac{a^2}{R^2}\right)^3\right], \tag{2.3}$$

$$b_0 = \alpha\left[3 - 2\frac{a^2}{R^2} + 3\beta\frac{a^2}{R^2}\left(2 + \frac{a^2}{R^2} + \frac{a^4}{R^4}\right)\right], \tag{2.4}$$

$$b_2 = \alpha a^4\left[1 + \beta\left(1 + 3\frac{a^2}{R^2}\right)\right], \tag{2.5}$$

$$b_3 = -2\alpha a^2\left[2\frac{a^2}{R^2} + 3\beta\left(1 + \frac{a^2}{R^2} + 2\frac{a^4}{R^4}\right)\right], \tag{2.6}$$

$$b_4 = -\alpha\beta\frac{a^2}{R^4}\left(3 + \frac{a^2}{R^2}\right), \tag{2.7}$$

and $b_1 = -2(1-\nu)b_3$. The parameters α and β are introduced as:

$$\alpha = \frac{\tau/4\mu}{(1 - a^2/R^2)(3 + a^2/R^2)}, \quad \beta = \frac{1 - 2a^2/R^2}{2(1 + 3a^4/R^4)}. \tag{2.8}$$

For large radius of the cylinder R, the b-constants become:

$$\begin{aligned} b_0 &= \frac{\tau}{4\mu}\left(1 + \frac{a^2}{R^2}\right), \\ b_3 &= -\frac{\tau}{4\mu}a^2\left(1 + \frac{a^2}{R^2}\right), \\ b_4 &= -\frac{\tau}{8\mu}\frac{a^2}{R^4}\left(1 - \frac{a^2}{R^2}\right), \end{aligned} \tag{2.9}$$

and $b_2 = 0$, neglecting the terms of higher order in a^2/R^2. Within the same order of accuracy:

$$\begin{aligned} \alpha &= \frac{\tau}{36\mu}\left(3 + 2\frac{a^2}{R^2}\right), \\ \beta &= \frac{1}{2}\left(1 - 2\frac{a^2}{R^2}\right). \end{aligned} \tag{2.10}$$

The last two sets of equations are used to obtain the asymptotic expressions for the displacement components at the outer surface of the cylinder, needed for the energy calculations in Section 4. These expressions are:

$$u_r^C(R,\theta) = \frac{\tau R}{2\mu}\left[1 + 2(1-\nu)\frac{a^2}{R^2} + O\left(\frac{a^4}{R^4}\right)\right]\sin 2\theta, \tag{2.11}$$

$$u_\theta^C(R,\theta) = \frac{\tau R}{2\mu}\left[1 + O\left(\frac{a^4}{R^4}\right)\right]\cos 2\theta. \tag{2.12}$$

3. SLIDING INHOMOGENEITY UNDER REMOTE SHEAR LOADING

Because of its significance in the analysis of inhomogeneous materials and fiber composites, in this section we consider a sliding circular inhomogeneity with isotropic elastic properties represented by the shear modulus and Poisson ratio (μ_1, ν_1), which is surrounded by a concentric circular cylinder with the elastic properties (μ_2, ν_2). The radius of the inhomogeneity is a, and the radius of the cylinder is R. Assume that the only loading is applied on the outer surface of the cylinder, and that it again corresponds to uniform state of shear stress $\sigma_{xy} = \tau$. The Papkovich–Neuber potentials can be taken in the general form given by Eqs. (1.1) and (1.4). The corresponding displacement and stresses are defined by Eqs. (1.5)–(1.14), with omitted terms proportional to γ. The following displacement and stress components are thus found. The displacements in the inhomogeneity are:

$$u_r^I = 2\left(b_0 + a_1 + 2\nu_1 a_2 r^2\right) r \sin 2\theta, \tag{3.1}$$

$$u_\theta^I = 2\left[b_0 + a_1 + (3-2\nu_1)a_2 r^2\right] r \cos 2\theta, \tag{3.2}$$

and the stresses:

$$\sigma_r^I = 4\mu_1\left(b_0 + a_1\right)\sin 2\theta, \tag{3.3}$$

$$\sigma_\theta^I = -4\mu_1\left(b_0 + a_1 + 6a_2r^2\right)\sin 2\theta, \tag{3.4}$$
$$\sigma_{r\theta}^I = 4\mu_1\left(b_0 + a_1 + 3a_2r^2\right)\cos 2\theta. \tag{3.5}$$

The displacement components in the cylinder are similarly:

$$u_r^C = 2\left[b_0 - b_2r^{-4} - (1-\nu_2)\,b_3r^{-2} + 2\nu_2b_4r^2\right] r\sin 2\theta, \tag{3.6}$$
$$u_\theta^C = 2\Big[b_0 + b_2r^{-4} - \frac{1}{2}(1-2\nu_2)\,b_3r^{-2} + (3-2\nu_2)\,b_4r^2\Big] r\cos 2\theta. \tag{3.7}$$

The stress components are:

$$\sigma_r^C = 4\mu_2\left(b_0 + 3b_2r^{-4} + b_3r^{-2}\right)\sin 2\theta, \tag{3.8}$$
$$\sigma_\theta^C = -4\mu_2\left(b_0 + 3b_2r^{-4} + 6b_4r^2\right)\sin 2\theta, \tag{3.9}$$
$$\sigma_{r\theta}^C = 4\mu_2\left(b_0 - 3b_2r^{-4} - \frac{1}{2}b_3r^{-2} + 3b_4r^2\right)\cos 2\theta, \tag{3.10}$$

provided that $b_1 = -2(1-\nu_2)b_3$. After a tedious calculation, the constants a's and b's are found to be:

$$a_1 = 6\alpha_1\left(1 - \frac{a^2}{R^2}\right) - \alpha_2\left(3 - 2\frac{a^2}{R^2}\right) + 3b_4\frac{R^4}{a^2c}\left[\frac{\mu_2}{\mu_1}\left(1-\frac{a^2}{R^2}\right)^3 + \frac{a^2}{R^2}\left(2 + \frac{a^2}{R^2} + \frac{a^4}{R^4}\right)\right], \tag{3.11}$$

$$a_2 = -2\alpha_1a^{-2}\left(1-\frac{a^2}{R^2}\right) - b_4\frac{\mu_2}{\mu_1}\frac{R^4}{a^4c}\left(1-\frac{a^2}{R^2}\right)^3, \tag{3.12}$$

$$b_0 = \alpha_2\left(3 - 2\frac{a^2}{R^2}\right) - 3b_4\frac{R^2}{c}\left(2 + \frac{a^2}{R^2} + \frac{a^4}{R^4}\right), \tag{3.13}$$

$$b_2 = \alpha_2a^4 - b_4\frac{R^4a^2}{c}\left(1 + 3\frac{a^2}{R^2}\right), \tag{3.14}$$

$$b_3 = -4\alpha_2\frac{a^4}{R^2} + 6b_4\frac{R^4}{c}\left(1 + \frac{a^2}{R^2} + 2\frac{a^4}{R^4}\right), \tag{3.15}$$

$$b_4 = \frac{\tau}{2d\mu_2}\frac{a^2/R^4}{1 - a^2/R^2}$$

$$\times \left\{ \left[1 - (3 - 2\nu_1) \frac{\mu_2}{\mu_1} \right] \left(1 - \frac{a^2}{R^2} \right) + 2(1 - \nu_2) \frac{a^2}{R^2} \right\}. \quad (3.16)$$

The parameters α_1 and α_2 are defined by the first part of Eq. (2.10), if the shear moduli μ_1 and μ_2 are there used, respectively. The parameters c and d are introduced by $c = 3 + a^2/R^2$, and

$$d = (5 - 6\nu_2) \left(1 + 3\frac{a^4}{R^4} \right) + (3 - 2\nu_2) \frac{a^2}{R^2} \left(3 + \frac{a^4}{R^4} \right) + (3 - 2\nu_1) \frac{\mu_2}{\mu_1} \left(1 - \frac{a^2}{R^2} \right)^3. \quad (3.17)$$

If $\mu_1 = \mu_2$ and $\nu_1 = \nu_2$, Eqs. (3.11)–(3.16) for the constants a's and b's reduce to those given in Section 2. If $R \to \infty$, the inhomogeneity is in an infinite matrix under remote shear loading at infinity. In this case, the constants a's and b's significantly simplify, since $\alpha_1 \to \tau/12\mu_1$, $\alpha_2 \to \tau/12\mu_2$, and:

$$R^4 b_4 \to \frac{\tau a^2}{2\mu_2} \frac{1 - (3 - 2\nu_1)\mu_2/\mu_1}{5 - 6\nu_2 + (3 - 2\nu_1)\mu_2/\mu_1}. \quad (3.18)$$

The explicit expressions for the corresponding stress and displacement fields in both the inhomogeneity and the matrix are given by Lubarda and Markenscoff (1999). For the rigid inhomogeneity, the relevant constants are:

$$b_0 = \frac{\tau}{4\mu_2}, \quad b_2 = \frac{\tau a^4}{4\mu_2} \frac{1 - 2\nu_2}{5 - 6\nu_2}, \quad b_3 = \frac{\tau a^2}{\mu_2} \frac{1}{5 - 6\nu_2}, \quad b_4 = 0. \quad (3.19)$$

For the void,

$$b_0 = \frac{\tau}{4\mu_2}, \quad b_2 = \frac{\tau a^4}{4\mu_2}, \quad b_3 = -\frac{\tau a^2}{\mu_2}, \quad b_4 = 0. \quad (3.20)$$

The stress field around the void does not depend on the Poisson ration of the matrix. Thus, from Eqs. (3.19) and (3.20), the stress field around the rigid inclusion in an artificial matrix having the Poisson ratio equal to 1 would be equal to the stress field around the void in an arbitrary real matrix. Observation of this kind was earlier made in the context of general plane elasticity by Dundurs (1989).

4. ENERGY CONSIDERATIONS

The elastic strain energy in the inclusion which has undergone a stress-free eigenstrain transformation ϵ_{ij}^* is equal to the work done on the inclusion to insert it back into the cylinder. This is

$$E^I = \frac{1}{2} \int_V \sigma_{ij}^I (\epsilon_{ij}^I - \epsilon_{ij}^*)\, \mathrm{d}V, \tag{4.1}$$

V being the volume of the inclusion. The strain energy in the cylinder is equal to the work done at the inner surface S by tractions produced there by the inserted inclusion, which is

$$E^C = -\frac{1}{2} \int_S \sigma_{ij}^I u_i^C n_j \, \mathrm{d}S, \tag{4.2}$$

where n_j are the components of the unit normal to S. The total strain energy, in the inclusion and the cylinder, is $E^T = E^I + E^C$, i.e.,

$$E^T = -\frac{1}{2} \int_V \sigma_{ij}^I \epsilon_{ij}^* \, \mathrm{d}V. \tag{4.3}$$

In the case of the bonded interface between the inclusion and the cylinder, this follows because $u_i^I = u_i^C$ at the interface, and in the case of the slipping interface, because $\sigma_{ij}^I n_j (u_i^I - u_i^C) = 0$, the traction vector being normal to the slip vector $u_i^I - u_i^C$ at the interface. In calculations, it is convenient to first calculate the total energy and the energy in the cylinder, and then the energy in the inclusion as their difference.

For example, in the case of the sliding inclusion in an infinitely extended matrix, the following expressions for the strain energies (per unit length in the z direction) are obtained for the shear eigenstrain $\epsilon_{xy}^* = \gamma$:

$$\begin{aligned} E_S^T &= \frac{3\mu\gamma^2}{8(1-\nu)}\, a^2\pi, \\ E_S^C &= \frac{3(5-6\nu)\mu\gamma^2}{64(1-\nu)^2}\, a^2\pi, \\ E_S^I &= \frac{3(3-2\nu)\mu\gamma^2}{64(1-\nu)^2}\, a^2\pi. \end{aligned} \tag{4.4}$$

The subscript S indicates the sliding interface. The energy in the matrix surrounding the inclusion is greater than the energy in the inclusion, more so less the material is compressible. For an incompressible material, $E_S^C = 2E_S^I$.

In the case of the inclusion with bonded interface, the energies are:

$$\begin{aligned} E_B^T &= \frac{\mu\gamma^2}{2(1-\nu)}\, a^2\pi, \\ E_B^C &= \frac{(3-4\nu)\mu\gamma^2}{8(1-\nu)^2}\, a^2\pi, \\ E_B^I &= \frac{\mu\gamma^2}{8(1-\nu)^2}\, a^2\pi. \end{aligned} \tag{4.5}$$

The energy in the matrix is again greater than in the inclusion, but in contrast to the slipping interface, more so more the material is compressible. For an incompressible material, $E_B^C = E_B^I$.

An appealing result follows by comparing the strain energies stored in the sliding and bonded inclusions. From the last of Eqs. (4.4) and (4.5), the difference in these energies is

$$E_S^I - E_B^I = \frac{(1-6\nu)\mu\gamma^2}{64(1-\nu)^2}\, a^2\pi. \tag{4.6}$$

Thus, $E_S^I < E_B^I$ if $\nu > 1/6$, and $E_S^I > E_B^I$ if $\nu < 1/6$. Hence, the shear stress relaxation at the interface actually increases the strain energy in the inclusion for very compressible materials ($\nu < 1/6$). Such an observation was first made in the case of an arbitrary but uniform eigenstrain by Lubarda and Markenscoff (1999). Note, however, that the total strain energy is always smaller in the case of the sliding inclusion. In fact, $E_S^T = (3/4)E_B^T$.

There is an analogous result that holds for the sliding inclusion under remote shear loading τ. The elastic strain energy in the sliding inclusion is

$$E_S^I = \frac{1}{2}\int_0^{2\pi} \sigma_r^I u_r^I a\, \mathrm{d}\theta = 4\mu(b_0 + a_1)(b_0 + a_1 + 2\nu a^2 a_2)a^2\pi, \tag{4.7}$$

where σ_r^I and u_r^I from Section 2 are both evaluated at $r = a$. The parameters a_1, a_2 and b_0 are defined by Eqs. (2.2)–(2.4). If the outer radius of the cylinder $R \to \infty$, these parameters become: $a_1 = \tau/8\mu$, $a_2 = -\tau a^{-2}/8\mu$ and $b_0 = \tau/4\mu$. Thus, in this case

$$E_S^I = \frac{3(3-2\nu)\tau^2}{16\mu}\, a^2\pi, \tag{4.8}$$

and

$$E_S^I - E_B^I = \frac{(1-6\nu)\tau^2}{16\mu}\, a^2\pi, \tag{4.9}$$

where $E_B^I = (\tau^2/2\mu)a^2\pi$. Consequently, the shear stress relaxation at the interface increases or decreases the elastic strain energy in the inclusion, again depending on the value of the Poisson ratio. The energy remains unchanged for $\nu = 1/6$.

The total strain energy, in the cylinder and in the sliding inclusion, is equal to the work done by the tractions at the outer surface of the cylinder on the corresponding displacement there, i.e.,

$$E_S^T = \frac{1}{2}\int_0^{2\pi} \left[\tau \sin 2\theta\, u_r^C(R,\theta) + \tau \cos 2\theta\, u_\theta^C(R,\theta)\right] R\, \mathrm{d}\theta. \tag{4.10}$$

Simple results hold in the limit as $R \to \infty$. By using the asymptotic expressions given by Eqs. (2.11) and (2.12) of Section 2, there follows

$$E_S^T - E_B^T = \frac{(1-\nu)\tau^2}{2\mu}\, a^2\pi, \tag{4.11}$$

where $E_B^T = (\tau^2/2\mu)R^2\pi$ (with infinitely extended radius R). This shows that the total strain energy increases by shear stress relaxation at the interface between the inclusion and the surrounding matrix. This is so because the whole system becomes more compliant in the presence of the frictionless interface, and the average deformation and thus the strain energy both increase.

An alternative derivation of the previous result is based on the potential energy consideration. If Π_S^T is the potential energy of the cylinder with the sliding inclusion (which is the strain energy E_S^T minus the load potential at the outer boundary of the cylinder), and if Π_B^T is the potential energy of the whole system without the slipping interface, the difference in two potential energies is equal to the work done by the shear stress relaxation on the slip discontinuity, $\Delta u_\theta = [(1-\nu)\tau/\mu]a\sin 2\theta$, at the interface. Specifically,

$$\Pi_S^T - \Pi_B^T = \frac{1}{2}\int_0^{2\pi}(-\tau\cos 2\theta)\Delta u_\theta\, a\,\mathrm{d}\theta = -\frac{(1-\nu)\tau^2}{2\mu}\,a^2\pi. \quad (4.12)$$

Since the value of the potential energy for an elastic system is equal to minus the value of the elastic strain energy of that system ($\Pi^T = -E^T$), Eq. (4.12) is evidently in accord with Eq. (4.11).

Implicitly contained in the described potential energy consideration is the following argument. The strain energy of the cylinder, without a slipping interface at $r = a$, is $E_B^T = (\tau^2/2\mu)R^2\pi$. When the slip is allowed at $r = a$, there is a change in the strain energy of the whole system due to the work done by the relaxing shear stress at the interface on the slip discontinuity there. Furthermore, there is a change in the strain energy due to the work done by the already applied traction at the remote boundary $r = R$ on additional displacements produced there by the shear stress relaxation at the interface $r = a$. Thus,

$$\begin{aligned} E_S^T - E_B^T = &\frac{1}{2}\int_0^{2\pi}(-\tau\cos 2\theta)\Delta u_\theta\, a\,\mathrm{d}\theta \\ &+\int_0^{2\pi}(\tau\sin 2\theta\,\delta u_r + \tau\cos 2\theta\,\delta u_\theta)R\,\mathrm{d}\theta, \end{aligned} \quad (4.13)$$

where

$$\delta u_r = u_r^S(R,\theta) - (\tau/2\mu)R\sin 2\theta,$$

and $\delta u_\theta = u_\theta^S(R,\theta) - (\tau/2\mu)R\cos 2\theta$. The first integral on the right-hand side of Eq. (4.13) is equal to $-(1-\nu)(\tau^2/2\mu)a^2\pi$, while the second integral is $(1-\nu)(\tau^2/\mu)a^2\pi$. Together, they yield the same result as given by Eq. (4.11).

We end this analysis by reporting two energy results for the circular inhomogeneity in an infinite matrix under remote uniform shear loading. For simplicity, incompressible materials are considered only. The difference between the strain energies in the sliding and bonded inhomogeneity is found to be

$$E_S^I - E_B^I = -\frac{1}{(1+\mu_2/\mu_1)^2}\frac{\tau^2}{2\mu_1}\,a^2\pi. \tag{4.14}$$

Thus, the energy in the sliding inhomogeneity is smaller than in the bonded inhomogeneity for all ratios of the shear moduli μ_2/μ_1. The total energy change in the considered system, associated with the transition from the bonded to the slipping interface between the matrix and the inhomogeneity, is

$$E_S^T - E_B^T = \frac{1}{1+\mu_2/\mu_1}\frac{\tau^2}{2\mu_2}\,a^2\pi. \tag{4.15}$$

Thus, slipping at the interface always increases the total energy. Energy expressions for arbitrary isotropic materials are listed by Lubarda and Markenscoff (1999).

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