ЦРНОГОРСКА АКАДЕМИЈА НАУКА И УМЈЕТНОСТИ
ГЛАСНИК ОДЈЕЉЕЊА ПРИРОДНИХ НАУКА, 15, 2003.
ЧЕРНОГОРСКАЯ АКАДЕМИЯ НАУК И ИСКУССТВ
ГЛАСНИК ОТДЕЛЕНИЯ ЕСТЕСТВЕННЫХ НАУК, 15, 2003.
THE MONTENEGRIN ACADEMY OF SCIENCES AND ARTS
GLASNIK OF THE SECTION OF NATURAL SCIENCES, 15, 2003.
UDK 621.879.32.3:519.67

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BENDING OF STRUCTURAL ELEMENTS OF THE HELICAL SHELL SHAPE

A b s t r a c t

In this paper a derivation of differential equation in displacement for helical shell whose parametric equations of the middle surface are: $x = \theta^1 \cos\theta^2$, $y = \theta^1 \sin\theta^2$, $z = k\theta^2$, $\theta^1, \theta^2 \equiv r, \varphi$ (r, φ polar coordinates), $\theta^1 \in (a, b), \theta^2 \in (-\infty, +\infty)$ is given. The shell is clamped on helix $\theta^1 = a$ and free on helix $\theta^2 = b$. The uniform pressure acts upon the shell. The restricted theory of shells is employed as a starting point for all derivations because we cannot use results of the classical shell theory. The reason for this is that the mixed coefficient of the second fundamental form of the surface is $B_{12} \neq 0$ i.e. coordinate lines aren't lines of curvature. We assume that the displacements in tangent plane of the middle surface u_1, u_2 can be neglected in comparison with displacement along normal u_3 and all derived kinematical measures aren't functions of θ^2. In special cases, for $k = 0$ and $k = \infty$ we obtain known equations for circular plate with hole and infinite length plate, respectively. The equation is solved numerically. These results are compared with those obtained bye FEA software code Pro/MECHANICA.

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SAVIJANJE KONSTRUKCIJSKIH ELEMENATA OBLIKA ZAVOJNE LJUSKE

Izvod

U ovom radu je dato izvođenje diferencijalne jednačine po pomjeranju za zavojnu ljusku, čije su parametarske jednačine srednje površi: $x = \theta^1 \cos\theta^2$, $y = \theta^1 \sin\theta^2$, $z = k\theta^2$, $\theta^1, \theta^2 \equiv r, \varphi$ (r, φ polarne koordinate), $\theta^1 \in (a, b), \theta^2 \in (-\infty, +\infty)$. Ljuska je ukliještena na zavojnici $\theta^1 = a$ i slobodna na zavojnici $\theta^2 = b$. Jednoliko raspoređeni pritisak djeluje na ljusku. Resta teorija ljuski se koristi kao polazna tačka za sva izvođenja zato što nismo mogli koristiti rezultate klasične teorije ljuski. Razlog za ovo leži u činjenici da je mješoviti koeficijent druge fundamentalne forme površi $B_{12} \neq 0$, tj. koordinatne linije nisu linije krivine. Predpostavili smo da se pomjeranja u tangentnoj ravni u_1, u_2 mogu zanemariti u poređenju sa pomjeranjem u pravcu normale u_3 i da izvedene deformacijske veličine nisu funkcija od θ^2. U specijalnim slučajevima za $k = 0$ i $k = \infty$ dobijamo poznate jednačine za kružnu ploču sa otvorom i ploču beskonačne dužine, respektivno. Diferencijalna jednačina je riješena numerički. Ovako dobijeni rezultati su upoređeni sa podacima dobijenim korišćenjem programa za MKE Pro/MECHANICA.

1. INTRODUCTION

Structural elements of the helicoidal shell shape are met at continual transport devices - helical transporters, bulidnigs machines, particularly machines for cleansing of snow, mining machines and so on. In thes paper is given a derivation of the differential equation in displacement for the helicoidal shell.

2. THE GEOMETRIC CHARACTERISTICS OF THE HELICOIDAL SURFACE

The parametric equations of the middle surface of helicoidal shell which is investigated are

$$\begin{gathered} \mathrm{x}_1 = \mathrm{x} = \theta^1 \cos\theta^2, \mathrm{x}_2 = \mathrm{y} = \theta^1 \sin\theta^2, \mathrm{x}_3 = \mathrm{z} = \mathrm{k}\theta^2, \\ \theta^1 \in (\mathrm{a}, \mathrm{b}), \theta^2 \in (-\infty, +\infty), \end{gathered} \tag{1}$$

where parameters θ^1, θ^2 are equivalent to polar coordinates r, φ.

The covariant base vectors $\mathbf{A}_\alpha$ are*

$$\begin{aligned} \mathbf{A}_1 &= \mathbf{R}_{,1} = \cos\theta^2\mathbf{i} + \sin\theta^2\mathbf{j} + 0\mathbf{k}, \\ \mathbf{A}_2 &= \mathbf{R}_{,2} = -\theta^1\sin\theta^2\mathbf{i} + \theta^1\cos\theta^2\mathbf{j} + \mathrm{k}\mathbf{k}, \end{aligned} \tag{2}$$

where $\mathbf{R}$ is position vector and comma denotes partial differentiation with respect to θ^α- coordinate. The unit normal $\mathbf{A}_3$ to the surface is

$$\mathbf{A}_3 = \frac{\mathbf{A}_1 \times \mathbf{A}_2}{|\mathbf{A}_1 \times \mathbf{A}_2|} = \frac{1}{\left[(\theta^1)^2 + \mathrm{k}^2\right]^{1/2}}\left[\mathrm{k}\sin\theta^2\mathbf{i} - \mathrm{k}\cos\theta^2\mathbf{j} + \theta^1\mathbf{k}\right]. \tag{3}$$

The coeficients (covariant) of the first fundamental form in case of helicoidal surface are

$$\begin{aligned} \mathrm{A}_{11} &= \mathbf{A}_1 \cdot \mathbf{A}_1 = 1, \\ \mathrm{A}_{12} = \mathrm{A}_{21} &= \mathbf{A}_1 \cdot \mathbf{A}_2 = 0, \\ \mathrm{A}_{22} &= \mathbf{A}_2 \cdot \mathbf{A}_2 = (\theta^1)^2 + \mathrm{k}^2. \end{aligned} \tag{4}$$

For the helicoidal surface, the covariant coeficients of the second fundamental form are

$$\begin{aligned} \mathrm{B}_{11} &= \mathbf{A}_3 \cdot \mathbf{A}_{1,1} = 0, \\ \mathrm{B}_{12} = \mathrm{B}_{21} &= \mathbf{A}_3 \cdot \mathbf{A}_{1,2} = -\frac{\mathrm{k}}{\left[(\theta^1)^2 + \mathrm{k}^2\right]^{1/2}}, \\ \mathrm{B}_{22} &= \mathbf{A}_3 \cdot \mathbf{A}_{2,2} = 0, \end{aligned} \tag{5}$$

since the mixed coeficients are

$$\mathrm{B}_1^1 = 0,\ \mathrm{B}_1^2 = -\frac{\mathrm{k}}{\left[(\theta^1)^2 + \mathrm{k}^2\right]^{3/2}},\ \mathrm{B}_2^1 = -\frac{\mathrm{k}}{\left[(\theta^1)^2 + \mathrm{k}^2\right]^{1/2}},\ \mathrm{B}_2^2 = 0. \tag{6}$$

The necessary and sufficient conditions that coordinate curves on surface be lines of curvature are $\mathrm{A}_{12} = 0$ and $\mathrm{B}_{12} = 0$. As the second condition is not satisfied then the generators and the helices are not lines of curvature, so that we must use appropriate relations in invariant (tensor) form. These relations are given in [1].

*All Greek indices have the range 1,2

3. THE BASIC KINEMATIC MEASURES

The basic kinematic measures will be derived according to so - called *restricted* theory which is a special case of the general theory of Cosserat surface. In the general theory, to every point of the middle surface od shell besides position vector is assigned deformable vector called *director* which can be rotated and linear deformed independently on deformation in that point. The relations of the restricted theory can be derived by considering the first and second fundamental form for thmed configuration of shell. We can express the basic kinematic measure in form

$$\mathrm{e}_{\alpha\beta} = \frac{1}{2}(\mathrm{u}_{\alpha|\beta} + \mathrm{u}_{\alpha|\beta}) - \mathrm{B}_{\alpha\beta}\mathrm{u}_3, \tag{7}$$

$$\chi_{\alpha\beta} = -\left[\mathrm{u}_{3|\alpha\beta} + \mathrm{B}^{\nu}_{\alpha|\beta}\mathrm{u}_\nu + \mathrm{B}^{\nu}_{\alpha}\mathrm{u}_{\nu|\alpha} - \mathrm{B}^{\nu}_{\beta}\mathrm{B}_{\alpha\nu}\mathrm{u}_3\right] = -\bar{\chi}_{\alpha\beta} = -\bar{\chi}_{\beta\alpha}, \tag{8}$$

where vertical bar denotes covariant derivative. We record here only the basic kinematic measures in physical components which are of interest in our analysis [2]

$$\begin{aligned}
e_{<12>} &= \frac{1}{2\left[(\theta^1)^2+k^2\right]^{1/2}}\frac{\partial u_{<1>}}{\partial\theta^2} + \frac{1}{2}\left[\frac{\partial u_{<2>}}{\partial\theta^1} - \frac{\theta^1}{(\theta^1)^2+k^2}u_{<2>}\right] + \frac{k}{(\theta^1)^2+k^2}u_3 \\
\bar{\chi}_{<11>} &= -\frac{2k}{(\theta^1)^2+k^2}\frac{\partial u_{<2>}}{\partial\theta^1} + \frac{2k\theta^1}{\left[(\theta^1)^2+k^2\right]^2}u_{<2>} + \frac{\partial^2 u_3}{\partial(\theta^1)^2} - \frac{k^2}{\left[(\theta^1)^2+k^2\right]^2}u_3 \\
\bar{\chi}_{<22>} &= -\frac{2k}{\left[(\theta^1)^2+k^2\right]^{3/2}}\frac{\partial u_{<1>}}{\partial\theta^2} + \frac{1}{(\theta^1)^2+k^2}\frac{\partial^2 u_3}{\partial(\theta^2)^2} + \frac{\theta^1}{(\theta^1)^2+k^2}\frac{\partial u_3}{\partial\theta^1} - \\
&\quad - \frac{k^2}{\left[(\theta^1)^2+k^2\right]^2}u_3
\end{aligned} \tag{9}$$

4. THE EQUATIONS OF EQUILIBRIUM

Taking into account the balance of linear momentum for shell are leading to the following equations

$$\mathrm{N}^{\alpha\beta}{}_{|\alpha} - \mathrm{B}^{\beta}_{\alpha}\mathrm{N}^{\alpha 3} + \rho_0\mathrm{F}^{\beta} = 0, \quad \mathrm{N}^{\alpha 3}{}_{|\alpha} - \mathrm{B}_{\alpha\beta}\mathrm{N}^{\alpha\beta} + \rho_0\mathrm{F}^3 = 0. \tag{10}$$

The remaing equations of equilibrium are being obtained by considering the balance of moment of momentum. They are:

$$\hat{\mathrm{M}}^{\beta\alpha}{}_{|\beta} - \mathrm{N}^{\alpha 3} = 0, \quad \bar{\varepsilon}_{\beta\alpha}\left[\mathrm{N}^{\alpha\beta} - \mathrm{B}^{\alpha}_{\gamma}\hat{\mathrm{M}}^{\gamma\beta}\right] = 0. \tag{11}$$

When the shell is loaded by uniform pressure along normal, that is $\rho_0 \mathrm{F}_{<3>} = -\mathrm{p}$, and clamped on $\theta^1 = \mathrm{a}$, we assume that all static quantities do not depend on coordinate θ^2.

Now we introduce the different set of components of contact force. By considering the sixth equation of equilibrium written in form $\mathrm{N}^{[\alpha\beta]} = \frac{1}{2}\left[\mathrm{B}^{\alpha}_{\gamma}\mathrm{M}^{\gamma\beta} - \mathrm{B}^{\beta}_{\gamma}\mathrm{M}^{\gamma\alpha}\right]$ ($\mathrm{N}^{[\alpha\beta]}$ is skew - symmetric part of $\mathrm{N}^{\alpha\beta}$), contact force components can be expressed as

$$\mathrm{N}^{\alpha\beta} = \hat{\mathrm{N}}^{\alpha\beta} + \mathrm{B}^{\alpha}_{\gamma}\mathrm{M}^{\gamma\beta} \tag{12}$$

where $\hat{\mathrm{N}}^{\alpha\beta}$ stands for certain symmetrical component. We assume that $\hat{\mathrm{M}}^{[\alpha\beta]} = 0$ in order to obtain a determinate theory. By substituting N^{13} and N^{23} from $(11)_{1,2}$ into (10) on account of (12), we obtain three equations. We only employ this one:

$$\begin{aligned}\frac{d^2\hat{M}_{<11>}}{d(\theta^1)^2} &+ \frac{2\theta^1}{(\theta^1)^2+k^2}\frac{d\hat{M}_{<11>}}{d\theta^1} - \frac{\theta^1}{(\theta^1)^2+k^2}\frac{d\hat{M}_{<22>}}{d\theta^1} - \\ &- \frac{2k^2}{\left[(\theta^1)^2+k^2\right]^2}\hat{M}_{<22>} - \frac{2k}{(\theta^1)^2+k^2}\hat{N}_{<12>} = p\end{aligned} \tag{13}$$

5. THE CONSTITUTIVE RELATIONS

The constitutive relations in restricted theory can be expressed as

$$\hat{\mathrm{N}}^{\alpha\beta} = \mathrm{C}\mathrm{H}^{\alpha\beta\gamma\delta}\mathrm{e}_{\gamma\delta}, \hat{\mathrm{M}}^{\alpha\beta} = -\mathrm{B}\mathrm{H}^{\alpha\beta\gamma\delta}\bar{\chi}_{\gamma\delta}, \tag{14}$$

$$\mathrm{H}^{\alpha\beta\gamma\delta} = \frac{1}{2}\left[\mathrm{A}^{\alpha\gamma}\mathrm{A}^{\beta\delta} + \mathrm{A}^{\alpha\delta}\mathrm{A}^{\beta\gamma} + \nu(2\mathrm{A}^{\alpha\beta}\mathrm{A}^{\gamma\delta} - \mathrm{A}^{\alpha\gamma}\mathrm{A}^{\beta\delta} - \mathrm{A}^{\alpha\delta}\mathrm{A}^{\beta\gamma})\right], \tag{15}$$

where C, B stand for extensional and flexural rigidity respectively. The physical sonstitutive equations for characteristic force and couples are

$$\begin{aligned}\hat{N}_{<12>} &= C(1-\nu)e^{<12>},\\ \hat{M}_{<11>} &= -B\left[\bar{\chi}_{<11>} + \nu\bar{\chi}_{<22>}\right],\\ \hat{M}_{<22>} &= -B\left[\bar{\chi}_{<22>} + \nu\bar{\chi}_{<11>}\right].\end{aligned} \tag{16}$$

6. THE DIFFRENTIAL EQUATION OF BENDING OF HELICOIDAL SHELL

As we consider only bending of the shell we supose that the displacements $u_{<1>}$, $u_{<2>}$ are small in comparison to $u_{<3>}$ which is in fact well known assumption made in the theory of shallow shells. Helicoidal shell is just a shallow one because of its small Gaussian curvature for $r >> 0$. By (9) and (16), after neglecting $\partial/\partial\varphi$, we can establish relations between force, couples and displacement u_3. By substituting these relations into (13), after lenghty but simple tion, we finally obtain the differential equation of bending of helicoidal shell

$$\frac{d^4u_3}{dr^4}+\frac{2r}{r^2+k^2}\frac{d^3u_3}{dr^3}-\frac{r^2+k^2(\nu-1)}{(r^2+k^2)^2}\frac{d^2u_3}{dr^2}+\frac{r\left[r^2+k^2(3\nu+4)\right]}{(r^2+k^2)^3}\frac{du_3}{dr}+$$
$$+\left[\frac{2k^2(\nu+1)(3k^2-8r^2)}{(r^2+k^2)^4}+\frac{24(1-\nu)k^2}{h^2(r^2+k^2)^2}\right]u_3+\frac{p}{B}=0, \tag{17}$$

where h is shell thicknes. In special case, for k =0, we have equation

$$\frac{d^4u_3}{dr^4}+\frac{2}{r}\frac{d^3u_3}{dr^3}-\frac{1}{r^2}\frac{d^2u_3}{dr^2}+\frac{1}{r^3}\frac{du_3}{dr}+\frac{p}{B}=0$$

or

$$\frac{1}{r}\frac{d}{dr}\left\{r\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{du_3}{dr}\right)\right]\right\}=-\frac{p}{B} \tag{18}$$

which is well known differential equation of bending of circular plate (with hole) under axial symmetric pressure. In other special case, for $k \to \infty$, we obtain

$$d^4u_3/dr^4=-p/B \tag{19}$$

which represent the differential equation of bending of infinite plate into cilindrical surface [3].

At the end of this chapter we must define boundary conditions. We need four boundary conditions as the differential equation is of fourth order. On the helix r =a they depend on displacement since on the helix r=b they depend on couple and force. As the displacements $u_{<\alpha>}$

have been neglected, the boundary conditions are

- on the helix r=a: $u_3 =_0 u_3, \partial u_3/\partial \nu_0 =_0 (\partial u_3/\partial \nu_0)$; (18)

- on the helix

- r=b: ${}_0G = \hat{M}_0^{\alpha\gamma} \nu_{\alpha 0} \nu_{\gamma},_0 P^3 =_0 \nu_\alpha (\hat{M}^{\beta\alpha}_{|\beta}) - \partial/\partial s_0 [\bar{\varepsilon}_{\beta\gamma} \hat{M}_0^{\alpha\beta} \nu_{\alpha 0} \nu^\gamma]$, (19)

where by index zero before appropriate quantity is denoted its values at boundary curve in undeformed state. The terms in boundary conditions have following meanings: ${}_0\nu_\alpha$ are the components of the outward unit normal to the boundary courve; $\partial/\partial\nu_0, \partial/\partial s_0$ denote the directional derivatives along the normal and the tangent; ${}_0G,_0 P^3$ represent couple and force along normal to the boundary curve.

In our case, the boundary curves are helices. Then the unit normal is ${}_0\nu = \mathbf{A}_1$, the derivative with respect to ν_0 coincides to derivative with respect to r, the derivative with respect to s_0 is proportional to derivative with respect to $\varphi (ds_0 = \sqrt{b^2 + k^2} d\varphi)$ so that it can be omitted. The helix r=a is clamped therefore ${}_0u_3 = 0$ and ${}_0(du_3/dr) = 0$. The helix r = b is free and so ${}_0G = 0$ and ${}_0P^3 = 0$. The boundary conditions can be expressed as follow:

- on the helix r=a: $u_3 = 0, \dfrac{du_3}{dr} = 0$; (20)

- on the helix r=b: $\hat{M}^{11} = 0, \hat{M}^{11}_{|1} + \hat{M}^{21}_{|2} = 0$

or after substituting:

$$
\begin{gathered}
\frac{d^2u_3}{dr^2} + \frac{\nu r}{r^2+k^2}\frac{du_3}{dr} - \frac{k^2(\nu+1}{(r^2+k^2)^2}u_3 = 0,\\
\frac{d^3u_3}{dr^3} + \frac{r}{r^2+k^2}\frac{d^2u_3}{dr^2} - \frac{1}{r^2+k^2}\frac{du_3}{dr} + \frac{4k^2(\nu+1)}{(r^2+k^2)^3}u_3 = 0.
\end{gathered}
\tag{21}
$$

7. THE NUMERICAL SOLUTION

The equations (17) cannot be solved in a closed form, so that we have to find out the solution numerically. For concrete values of quantities B = 1.172 [Nmm], p = 0.15 [N/mm^2], ν = 0.3, a = 130 [mm], b = 200 [mm], H = 140 [mm], k = H/2π, h = 4 [mm] maximum value of the function u_3 is for r = b and it equals to -0.4105 [mm]. We compared this result with one obtained by FEA software code Pro/MECHANICA. It is modelled helicoidal shell with 3 spirals and the same parameters as previous. The points on the helix r = b sufficiently distant from the generators $\varphi = \pm 3\pi$ have the same value of displacement u_3 equals to -0.4124 [mm].

The obtained and verified results in the paper give the opportunity for automatic design of structural elements of the helicoidal shell shape.

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