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# ON A CONTINUOUS GRADIENT-TYPE METHODS FOR SOLVING QUASI-VARIATIONAL INEQULITIES

*Abstract*

In this paper we study continuous gradient-type method for solving quasi-variational inequlities and establish sufficient conditions for the convergence of the proposed methods.

*Key words: continuous method, gradient-type method, quasi-variational inequlities*

# O NEPREKIDNIM METODAMA GRADIJENTNOG TIPA ZA RJEŠAVANJE KVAZIVARIJACIONIH NEJEDNAKOSTI

*Sažetak*

U radu se izučavaju neki neprekidni metodi gradijentnog tipa za rješavanje kvazivarijacionih nejednakosti i formulišu dovoljni uslovi za konvergenciju tih metoda.

*Ključne riječi: neprekidni metod, metod gradijentnog tipa, kvazivarijacione nejednakosti*

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## 1. INTRODUCTION

The theory, as well as solution methods of the variational inequality, have been well documented in the literature. Quasi-variational inequalities can be used to formulate the generalized Nash game in which each player strategy set depend on the other players strategies, not only his payoff function. The quasi-variational inequalities have recently attracted growing attention in relation to game theory. Methods for solving quasi-variational inequalities have been studied by Nesterov in [4]. In this paper, we present one of continuous gradient-type method for solving quasi-variational inequalities, which iterative method is presented in [4]. Note that similar method is presented in [5], but the conditions of convergence are different.

Let we denote by $H$ a Hilbert space. By $\pi_C(x)$ we denote the Euclidian projection of point $x$ onto the set $C$.

Let $C : H \to 2^H$ be a set-valued mapping with nonempty closed and convex values. Consider a continuous operator $F(x) : C \to H$, which is *strongly monotone*

$$\langle F(x) - F(y), x - y\rangle \geq \mu\|x - y\|^2, \quad \forall\, x, y \in C, \tag{1.1}$$

and *Lipschitz continuous* on $C$

$$\|F(x) - F(y)\| \leq L\|x - y\|, \quad \forall\, x, y \in C. \tag{1.2}$$

The constant $\mu \geq 0$ is called the parameter of strong monotonicity of operator $F$. If $\mu = 0$, then $F$ is a monotone operator. In what follows, we always assume $\mu > 0$. The constant $L$ is called Lipschitz constant.

The problem of our interest is the following quasi-variational inequality (QVI):

$$\text{Find } x_* \in C(x_*) \text{ such that } \langle F(x_*), y - x_*\rangle \geq 0, \ \forall y \in C(x_*). \tag{1.3}$$

In what follows, we will propose one continuous method for solving this problem and we establish some sufficient condition of the convergence of the proposed method. Since this method relies on the iterative gradient-type method, described in [4], we explore it here.

## 2. CONTINUOUS GRADIENT METHOD

The following theorem (see [4]) provides the necessary and sufficient conditions for the existence of solutions of problem (1.3).

**Theorem 1.** *Suppose that the following assumptions hold:*

*(a) Operator $F$ is Lipschitz continuous and strongly monotone on $H$ whit constants $L$ and $\mu > 0$ respectively.*

*(b) There exists $\alpha < \frac{\mu^2}{L(L+\sqrt{L^2-\mu^2})}$ such that*

$$\|\pi_{C(x)}(z) - \pi_{C(y)}(z)\| \le \alpha\|x - y\|, \quad \forall\ x, y, z \in H. \tag{2.1}$$

*Then the problem (1.3) has a unique solution.*

Under these assumptions, a problem (1.3) can be solved by a standard gradient method:

$$x_{k+1} = \pi_{C(x_k)}(x_k - \lambda F(x_k)),\ k \ge 0 \tag{2.2}$$

The following theorem has been proved in [4]:

**Theorem 2.** *If operator $F$ is strongly monotone and Lipschitz continuous with constants $L$ and $\mu$, and multifunction $C(x)$ satisfies condition (2.1) with $\alpha < \frac{\mu^2}{L(L+\sqrt{L^2-\mu^2})}$, then the gradient method (2.2) with optimal stepsize $\lambda = \frac{\mu}{L^2}$ converges to the unique solution of problem (1.3) with the following rate*

$$\|x_k - x_*\| \le \exp\left\{-k\left(\frac{1}{\gamma(\gamma + \sqrt{\gamma^2 - 1})} - \alpha\right)\right\}\|x_0 - x_*\|. \tag{2.3}$$

Note, we have seen that quasi-variational inequality (1.3) is solvable by gradient scheme (2.2) only if the variation rate of the feasible set $C(x)$ is very small.

For solving problem (1.3), one can use continuous gradient-type method:

$$x'(t) + x(t) = \pi_{C(x(t))}[x(t) - \lambda(t)F(x(t))], \quad t > 0,\ x(0) = x_0, \tag{2.4}$$

where $x_0$ is a given initial point.We will prove convergence of this method and convergance rate.

**Theorem 3.** *If operator $F$ is strongly monotone and Lipschitz continuous with constants $L$ and $\mu$, and multifunction $C(x)$ satisfies condition (2.1) with $\alpha < \frac{\mu^2}{L(L+\sqrt{L^2-\mu^2})}$, then the continuous gradient method (2.4) with parameter $\frac{\mu-\sqrt{\mu^2-L^2(2\alpha-\alpha^2)}}{L^2} \leq \lambda(t) \leq \frac{\mu+\sqrt{\mu^2-L^2(2\alpha-\alpha^2)}}{L^2}$ converges to the unique solution of problem (1.3) with the following rate*

$$\|x(t)-x_*\|^2 \leq \exp\{-a_0(t-t_0)\}\|x_0-x_*\|^2,$$

*where*

$$a(t) = 1-\left(\alpha+\sqrt{1-2\lambda(t)\mu+\lambda^2(t)L^2}\right)^2 \geq a_0 > 0.$$

*Proof.* We will use Lyapunov function

$$V(t) = \frac{1}{2}\|x(t)-x_*\|^2, \quad V(t_0) = \frac{1}{2}\|x_0-x_*\|^2 = V_0,$$

and then it is enough to prove following statement

$$V(t) \to 0 \text{ as } t \to \infty.$$

Consider derivative of Lyapunov function:

$$V'(t) = \langle x(t)-x_*, x'(t)\rangle.$$

Since $x_* = \pi_{C(x_*)}[x_*-\lambda(t)F(x_*)]$, we have

$$\begin{aligned} V'(t) &= \langle x(t)-\pi_{C(x_*)}[x_*-\lambda(t)F(x_*)], \pi_{C(x(t))}[x(t)-\lambda(t)F(x(t))]-x(t)\rangle \\ &= \frac{1}{2}\|\pi_{C(x(t))}[x(t)-\lambda(t)F(x(t))]-\pi_{C(x_*)}[x_*-\lambda(t)F(x_*)]\|^2 \\ &\quad -\frac{1}{2}\|\pi_{C(x(t))}[x(t)-\lambda(t)F(x(t))]-x(t)\|^2 \\ &\quad -\frac{1}{2}\|x(t)-\pi_{C(x_*)}[x_*-\lambda(t)F(x_*)]\|^2. \end{aligned} \tag{2.5}$$

Now, we are going to estimate some of the terms in (2.5):

$$\|\pi_{C(x(t))}[x(t)-\lambda(t)F(x(t))]-\pi_{C(x_*)}[x_*-\lambda(t)F(x_*)]\|$$

$$\begin{aligned}
&\le \|\pi_{C(x(t))}[x(t)-\lambda(t)F(x(t))]-\pi_{C(x_*)}[x(t)-\lambda(t)F(x(t))]\| \\
&\quad +\|\pi_{C(x_*)}[x(t)-\lambda(t)F(x(t))]-\pi_{C(x_*)}[x_*-\lambda(t)F(x_*)]\| \\
&\le \alpha\|x(t)-x_*\|+\|\pi_{C(x_*)}[x(t)-\lambda(t)F(x(t))] \\
&\quad -\pi_{C(x_*)}[x_*-\lambda(t)F(x_*)]\|. \qquad (2.6)
\end{aligned}$$

Since $F$ is strongly monotone and Lipschitz continuous, we have

$$\begin{aligned}
&\|\pi_{C(x_*)}[x(t)-\lambda(t)F(x(t))]-\pi_{C(x_*)}[x_*-\lambda(t)F(x_*)]\|^2 \\
\le\ &\|[x(t)-\lambda(t)F(x(t))]-[x_*-\lambda(t)F(x_*)]\|^2 \\
=\ &\|x(t)-x_*\|^2-2\lambda(t)\mu\langle F(x(t))-F(x_*),x(t-x_*)\rangle \\
&+\lambda^2(t)\|F(x(t))-F(x_*)\|^2 \\
\le\ &\|x(t)-x_*\|^2-2\lambda(t)\mu\|x(t)-x_*\|^2+\lambda^2(t)L^2\|x(t)-x_*\|^2 \\
=\ &\left(1-2\lambda(t)\mu+\lambda^2(t)L^2\right)\|x(t)-x_*\|^2. \qquad (2.7)
\end{aligned}$$

Combining (2.6) and (2.7), we have

$$\begin{aligned}
&\|\pi_{C(x(t))}[x(t)-\lambda(t)F(x(t))]-\pi_{C(x_*)}[x_*-\lambda(t)F(x_*)]\|^2 \\
\le\ &\left(\alpha+\sqrt{1-2\lambda\mu+\lambda^2L^2}\right)^2\|x(t)-x_*\|^2.
\end{aligned}$$

If we combine this with condition $x_*=\pi_{C(x_*)}[x_*-\lambda(t)F(x_*)]$, we get

$$V'(t)\le\frac{1}{2}\left[\left(\alpha+\sqrt{1-2\lambda(t)\mu+\lambda^2(t)L^2}\right)^2-1\right]\|x(t)-x_*\|^2,$$

i.e.

$$V'(t)\le\left[\left(\alpha+\sqrt{1-2\lambda(t)\mu+\lambda^2(t)L^2}\right)^2-1\right]V(t).$$

By condition $0<\alpha<\frac{\mu^2}{L(L+\sqrt{L^2-\mu^2})}$ for $\frac{\mu-\sqrt{\mu^2-L^2(2\alpha-\alpha^2)}}{L^2}\le\lambda(t)\le\frac{\mu+\sqrt{\mu^2-L^2(2\alpha-\alpha^2)}}{L^2}$, the following estimate has a place

$$\alpha+\sqrt{1-2\lambda(t)\mu+\lambda^2(t)L^2}<1.$$

where $a(t) = 1 - \left(\alpha + \sqrt{1 - 2\lambda(t)\mu + \lambda^2(t)L^2}\right)^2 \geq a_0 > 0$.

Hence,

$$V(t) \leq V_0 \exp\{-a_0(t - t_0)\},$$

so $V(t) \to 0$ as $t \to \infty$, i.e. $x(t)$ converges exponentially towards to $x_*$ with rate $a_0$. Thus, we have proved the theorem.

## 4. METHOD FOR SET C(x)=c(x)+C

Now we mention that assumption (2.1) is a kind of strengthening of the contraction property for multifunction $C(x)$. An example (see [4]) of such mapping is

$$C(x) := c(x) + \bar{C}, \tag{4.1}$$

where function $c(x)$ is Lipschitz continuous with Lipschitz constant $l = \alpha$ from condition (2.1) and $\bar{C}$ is a closed convex set.

In this case, the corresponding iterative gradient method for solving quasi-variational inequalities (1.3) has the form

$$x_{k+1} = c(x_k) + \pi_{\bar{C}}[x_k - c(x_k) - \lambda F(x_k)], \ k \geq 0$$

and continuous gradient method for solving quasi-variational inequalities has the form

$$x'(t)+x(t) = c(x(t))+\pi_{\bar{C}}[x(t)-c(x(t))-\lambda(t)F(x(t))], \ t \geq 0, \ x(0) = x_0. \tag{4.2}$$

Let us remark that the relation (4.2) is equivalent to the following variational inequality

$$\langle x'(t)+\lambda(t)F(x(t)), x'(t)+x(t)-c(x(t))-z\rangle \geq 0, \ \forall z \in \bar{C}, t > 0. \tag{4.3}$$

Setting $z = x_* - c(x_*) \in \bar{C}$ in (4.3), $y = c(x_*) + x'(t) + x(t) - c(x(t)) \in C(x_*)$ in (1.3) and multiplying (1.3) by $\lambda(t) > 0$, in the sum of the obtained inequalities we will get

$$\left\langle x'(t), x'(t) + x(t) + c(x_*) - c(x(t)) - x_*\right\rangle$$

$$\leq \quad \lambda(t)\langle F(x(t)) - F(x_*), x_* - x'(t) - x(t) + c(x(t) - c(x_*))\rangle$$

i.e.

$$\begin{aligned} & \langle x'(t), x'(t)\rangle + \langle x'(t), x(t) - x_*\rangle + \langle x'(t), c(x_*) - c(x(t))\rangle \\ \leq \; & \lambda(t)\langle F(x(t)) - F(x_*), -x'(t)\rangle + \lambda(t)\langle F(x(t)) - F(x_*), x_* - x(t)\rangle \\ & +\lambda(t)\langle F(x(t)) - F(x_*), c(x(t) - c(x_*))\rangle. \end{aligned} \tag{4.4}$$

Now, we are going to estimate some of the terms in (4.4). We will use the following equation

$$\langle x'(t), x(t) - x_*\rangle = \frac{1}{2}\frac{d}{dt}\|x(t) - x_*\|^2. \tag{4.5}$$

Since $c$ is Lipschitz continuous, we have

$$\begin{aligned} & \langle x'(t), c(x_*) - c(x(t))\rangle \geq -\|x'(t)\| \cdot \|c(x_*) - c(x(t))\| \geq \\ & -\frac{1}{2}\|x'(t)\|^2 - \frac{1}{2}\|c(x_*) - c(x(t))\|^2 \geq -\frac{1}{2}\|x'(t)\|^2 - \frac{1}{2}l^2\|x(t) - x_*\|. \end{aligned}$$

Note that Lipschitz continuous and strongly monotone mapping (with Lipschitz constant $L$ and parameter of strongly monotonicity $\mu$) satisfies the following inequality (see [6], p.180)

$$\|F(u) - F(v)\|^2 \leq (L + \mu)\langle F(u) - F(v), u - v\rangle - L\mu\|u - v\|^2. \tag{4.6}$$

If we combine this property of $F$ with inequality $2ab \leq a^2 + b^2$, we get

$$\begin{aligned} & \lambda(t)\langle F(x(t)) - F(x_*), -x'(t)\rangle \\ \leq \; & \frac{\lambda^2(t)}{2}\|F(x(t)) - F(x_*)\|^2 + \frac{1}{2}\|x'(t)\|^2 \\ \leq \; & \frac{\lambda^2(t)}{2}\left((L + \mu)\langle F(x(t)) - F(x_*), x(t) - x_*\rangle - L\mu\|x(t) - x_*\|^2\right) \\ & +\frac{1}{2}\|x'(t)\|^2. \end{aligned}$$

Since $F$ and $c$ are Lipschitz continuous with Lipschitz constants $L$ and $l$, respectively, next inequality holds

$$\langle F(x(t)) - F(x_*), c(x_*) - c(x(t)))\rangle \leq Ll\|x(t) - x_*\|^2.$$

Now, inequality (4.4), multiplying by 2 becomes

$$\begin{aligned}
& 2\|x'(t)\|^2 + \frac{d}{dt}\|x(t) - x_*\|^2 - \|x'(t)\|^2 - l^2\|x(t) - x_*\|^2 \\
\leq\ & \lambda^2(t)\left((L+\mu)\langle F(x(t)) - F(x_*), x(t) - x_*\rangle - L\mu\|x(t) - x_*\|^2\right) \\
& + \|x'(t)\|^2 - 2\lambda(t)\langle F(x(t)) - F(x_*), x(t) - x_*\rangle \\
& + 2lL\lambda(t)\|x(t) - x_*\|^2,
\end{aligned}$$

i.e.

$$\begin{aligned}
& \frac{d}{dt}\|x(t) - x_*\|^2 \leq \left(l^2 - \lambda^2(t)L\mu + 2lL\lambda(t)\right)\|x(t) - x_*\|^2 + \\
& \lambda(t)\left(2 - \lambda(t)(L+\mu)\right)\langle F(x(t)) - F(x_*), x_* - x(t)\rangle.
\end{aligned}$$

Let us suppose that

$$2 - \lambda(t)(L+\mu) > 0,$$

then bacause of (1.1), it follows

$$\begin{aligned}
& \frac{d}{dt}\|x(t) - x_*\|^2 \leq \left(l^2 - \lambda^2(t)L\mu + 2lL\lambda(t)\right)\|x(t) - x_*\|^2 \\
& -\mu\lambda(t)\left(2 - \lambda(t)(L+\mu)\right)\|x(t) - x_*\|^2,
\end{aligned}$$

hence

$$\frac{d}{dt}\|x(t) - x_*\|^2 \leq -A(t)\|x(t) - x_*\|^2,$$

where

$$A(t) = \left(-l^2 - 2lL\lambda(t) - \mu^2\lambda^2(t) + 2\mu\lambda(t)\right) \geq A_0 > 0.$$

Finally, we have inequality

$$\frac{dV(t)}{dt} \leq -A_0 V(t).$$

This yields that

$$\|x(t) - x_*\|^2 \leq \|x_0 - x_*\|^2 \exp\{-A_0(t - t_0)\}.$$

## 5. SECOND-ORDER GRADIENT-TYPE METHOD

In previous sections we discussed first-order methods for solving quasi-variational inequalities. Here, we will describe a continuous second-order method for solving QVI.

The method is based on the differential equation

$$\beta(t)x'' + x' + x = \Pi_{C(x)}(x - \lambda(t)F(x)), \quad \beta(t) > 0, \lambda(t) > 0. \tag{5.1}$$

where $x = x(t)$. Instead of $\beta(t)$ and $\lambda(t)$ we will write $\beta$ and $\lambda$, respectively. If the sets $C(x)$ are given as in (4.1), this method has the form

$$\beta x'' + x' + x = c(x) + \Pi_{\bar{C}}(x - \lambda F(x) - c(x)), \quad \beta > 0, \lambda > 0.$$

This relation is equivalent to the following variational inequality

$$\big\langle \beta x'' + x' + \lambda F(x), z - \beta x'' - x' - x + c(x)\big\rangle \geq 0, \forall z \in \bar{C}. \tag{5.2}$$

Setting $z = x_* - c(x_*) \in \bar{C}$ in (5.2), $y = \beta x'' + x' + x + c(x_*) - c(x) \in c(x_*) + \bar{C}$ in (1.3) and multiplying (1.3) by $\lambda > 0$, in the sum of the obtained inequalities we will obtain

$$\big\langle \beta x'' + x', \beta x'' + x' + x - x_* + c(x_*) - c(x)\big\rangle \leq$$
$$\lambda\langle F(x) - F(x_*), x_* - \beta x'' - x' - x - c(x_*) + c(x)\rangle.$$

Using $ab \leq a^2/2 + b^2/2$, we can write

$$\begin{aligned} & \|\beta x'' + x'\|^2 + \langle \beta x'' + x', x - x_*\rangle + \langle \beta x'' + x', c(x) - c(x_*)\rangle \leq \\ \leq\ & \frac{1}{2}\|\beta x'' + x'\|^2 + \frac{\lambda^2}{2}\|F(x) - F(x_*)\|^2 + \lambda\langle F(x) - F(x_*), x_* - x\rangle \\ +\ & \lambda\langle F(x) - F(x_*), c(x) - c(x_*)\rangle. \end{aligned} \tag{5.3}$$

From here, multiplying by 2, because of the strong convexity, Lipschitz continuity and conditions (4.6) we get the estimate

$$\|\beta x'' + x'\|^2 + 2\langle \beta x'' + x', x - x_*\rangle - l\|\beta x'' + x'\|^2 - l\|x - x_*\|^2 \leq$$

$$\lambda[2-(L+\mu)\lambda]\langle F(x)-F(x_*), x_*-x\rangle - L\mu\lambda^2\|x-x_*\|^2$$
$$+\lambda lL\|x-x_*\|^2 \le 0,\ t\ge 0$$

Suppose the condition for choice of the parameter $\lambda$

$$2-(L+\mu)\lambda>0,$$

and having in mind strong monotonicity of $F$, it follows

$$(1-l)\|\beta x''+x'\|^2+2\langle\beta x''+x', x-x_*\rangle \le$$
$$(l+\lambda lL-\mu\lambda[2-\mu\lambda])\,\|x-x_*\|^2. \tag{5.4}$$

Now we use equalities (4.5) and

$$\langle x''(t), x(t)-x_*\rangle = \frac{1}{2}\frac{d^2}{dt^2}\|x(t)-x_*\|^2-\|x'(t)\|^2, \tag{5.5}$$

and inequality (5.4) can be written as

$$(1-l)\beta^2\|x''\|^2+[1-l-2\beta]\|x'\|^2+(1-l)\beta\frac{d}{dt}\left(\|x'\|^2\right)+$$
$$\beta\frac{d^2}{dt^2}\left(\|x-x_*\|^2\right)+\frac{d}{dt}\left(\|x-x_*\|^2\right)+A(t)\|x-x_*\|^2\le 0, \tag{5.6}$$

where

$$A(t)=\lambda(t)\mu(2-\lambda(t)\mu)-l-lL\lambda(t)>0,$$

for $l<\min\left\{\frac{\mu}{L},\ \frac{2\mu}{L^2}\left(L+\mu-\sqrt{\mu(2L+\mu)}\right)\right\}$.

Let be $h(t)=\exp\left\{\int_0^t b(s)ds\right\}$, where $b(s)=\frac{1}{\beta}\left(1-\sqrt{1-4A(s)\beta}\right)$. If we inequality (5.6) multiply by $h(t)$ and integrating on segment $[0,t]$, for $\beta=const$ we get

$$(1-l)\beta^2\int_0^t h(s)\|x''(s)\|^2ds+(1-l)\beta h(t)\|x'(t)\|^2$$
$$+\ \beta h(t)\frac{d}{dt}\left(\|x(t)-x_*\|^2\right)+h(t)\left(1-\beta b(t)\right)\|x(t)-x_*\|^2$$
$$+\ \int_0^t h(s)\left(1-l-2\beta-\beta b(s)\right)\|x'(s)\|^2ds$$

$$
\begin{aligned}
&+ \int_0^t h(s)\left(\beta(b^2(s)+b'(s)) - b(s) + A(s)\right)\|x(s)-x_*\|^2 ds \\
&\le C_0,\ t \ge 0, \qquad (5.7)
\end{aligned}
$$

where $C_0 = (1-l)\beta h(0)\|x'(0)\|^2 + h(0)\left(1-\beta b(0)\right)\|x(0)-x_*\|^2$. All the integrals on the left side of the inequality are nonnegative, so we have

$$
\begin{aligned}
&(1-l)\beta h(t)\|x'(t)\|^2 + \beta h(t)\frac{d}{dt}\left(\|x(t)-x_*\|^2\right) \\
&+ \left(1-\beta b(t)\right) h(t)\|x(t)-x_*\|^2 \le C_0,\ t \ge 0. \qquad (5.8)
\end{aligned}
$$

Note that the first term on the left side of inequality (5.8) is nonnegative and $1-\beta b(t) > 0$, for $t \ge 0$. Whence, multiplying inequality (5.8) by $(\beta h(t))^{-1}$ finally we get linear differential inequality

$$
\frac{d}{dt}\left(\|x(t)-x_*\|^2\right) + \beta^{-1}(1-\beta b(t))\|x-x_*\|^2 \le C_0\left(\beta h(t)\right)^{-1},\ t \ge 0.
$$

If we multiply this inequality by

$$
H(T) = \exp\left\{\int_0^t f(s)ds\right\},\ \text{where } f(s) = \beta^{-1}(1-\beta b(s)),
$$

it will have the following form

$$
\frac{d}{dt}\left(\|x(t)-x_*\|^2 H(t)\right) \le C_0 H(t)\left(\beta h(t)\right)^{-1},\ t \ge 0.
$$

From this inequality, integrating on segment $[0,t]$, we get

$$
\|x(t)-x_*\|^2 H(t) \le \|x(0)-x_*\|^2 + C_0 \int_0^t H(s)(\beta h(s))^{-1} ds,\ t \ge 0.
$$

Whence

$$
\|x(t)-x_*\|^2 \le \|x(0)-x_*\|^2 H^{-1}(t) + C_0\rho(t)h^{-1}(t),\ t \ge 0,
$$

where $\rho(t) = h(t)H^{-1}(t)\int_0^t H(s)(\beta h(s))^{-1}ds$.

Nedić (see [4]) has considered and studied similar method and proved that $\lim_{n\to\infty} h(t) = \lim_{n\to\infty} H(t) = +\infty$ and $\lim_{n\to\infty}\rho(t) = const$. Hence, we have proved the following theorem.

**Theorem 4.** *Let operator $F$ be strongly monotone and Lipschitz continuous with constants $L$ and $\mu$. Let multifunction $C(x)$ be given as $C(x) = c(x) + \bar{C}$, where function $c(x)$ is Lipschitz continuous with constant $l$ such that $l < \min\left\{\frac{\mu}{L}, \frac{2\mu}{L^2}\left(L+\mu-\sqrt{\mu(2L+\mu)}\right)\right\}$ and $\bar{C}$ is a closed convex set. If parameters of method $\lambda(t) \in C^1[0,+\infty)$ and $\beta$ satisfies conditions*

$$\begin{aligned}
&\alpha(t) > 0,\ \beta > 0,\ \alpha'(t) \le 0, \quad t \ge 0; \quad \alpha(0) < 2(L+\mu)^{-1},\\
&\lim_{t\to\infty} \alpha(t) = \alpha_\infty > 0, \quad \sqrt{1-4A(0)\beta} \ge 4\beta - 1,\\
&1 - 4\mu\alpha_\infty\beta(2-\mu\alpha_\infty) - 4l\beta(1+L\alpha_\infty) > 0,
\end{aligned}$$

*then, the second-order gradient type method (5.1) converges to the unique solution of problem (1.3) with the following rate*

$$\begin{aligned}
\|x(t)-x_*\|^2 &\le \|x(0)-x_*\|^2 \exp\left\{-\int_0^t f(s)ds\right\}\\
&+ C_0\rho(t)\exp\left\{-\int_0^t b(s)ds\right\},
\end{aligned}$$

*where*

$$f(t) = \frac{1}{2\beta}\left(1+\sqrt{1-4A(t)\beta}\right) = \beta^{-1}(1-\beta b(t)) > 0,$$

$$C_0 = (1-l)\beta h(0)\|x'(0)\|^2 + h(0)\left(1-\beta b(0)\right)\|x(0)-x_*\|^2,$$

$$\rho(t) = h(t)H^{-1}(t)\int_0^t H(s)(\beta h(s))^{-1}ds,$$

$$b(t) = \frac{1}{\beta}\left(1-\sqrt{1-4A(t)\beta}\right),$$

$$A(t) = \lambda(t)\mu(2-\lambda(t)\mu) - l - lL\lambda(t) > 0.$$

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