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NEW S-CLASS OF TIME-FREQUENCY DISTRIBUTIONS

Abstract

A new general class of distributions (S-class of distributions) for time-frequency signal analysis is proposed. This class is derived by generalizing the rescently defined X-distribution. All the known and widely used distributions belong to this class (Spectrogram-Short time Fourier transform, Wigner distribution, Rihaczek distribution, Choi-Wiliams distribution, Cohen class of distributions, . . .). Some particular, new distributions, belonging to this class are introduced. It is possible to define the S-counterpart distribution for each known distribution from the Cohen class, such that some of the performances may be improved. This class of distributions may be treated as a variant of the author's L-class of distributions, but it may satisfy unbiased energy condition, time marginal, as well as the frequency marginal in the case of asymptotic signals. The presented theory is illustrated by examples.

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1. INTRODUCTION

Time-frequency analysis has attracted attention of many researchers. The main challenge in this area lies in the fact that many fundamental questions are still waiting for viable answers. The whole variety of tools for time-frequency analysis, mainly rendered in the form of energy distributions in the time-frequency plane, has been proposed (for a complete list, see the review papers [1, 2] and references therein). The oldest method for time-frequency signal analysis is based on the Fourier transform (its short time variant). It is a linear signal transformation. Many performances of the signal's representation may be improved using quadratic distributions. The first quadratic representation was based on the Wigner distribution. Afterwards, many other quadratic distributions have been defined. Cohen has shown that all shiftcovariant quadratic time-frequency distributions are just special cases of a general class of distributions, obtained for a particular choice of an arbitrary function (kernel) [6]. Out of the Cohen class, the Wigner distribution is the only one (with signal independent kernel) which produces the ideal energy concentration along instantaneous frequency for the linear frequency modulated signals, [10, 11, 12].

In order to improve the concentration of signal's energy, when the instantaneous frequency is polynomial function of time, the Polynomial Wigner distribution are proposed, [13, 14, 15]. A similar idea for improving the distribution concentration of the signal whose phase is polynomial up to the fourth order, was presented in 18. In order to improve distribution concentration for a signal with an arbitraty nonlinear instantaneous frequency, the L-Wigner distribution was proposed and studied in [11, 12, 18, 19]. This distribution is generalized to the L-class of distributions in [12, 38]. The Polynomial Wigner distribution, as well as the L-Wigner distributions, are closely related to the time-varying higher odder spectra, [19, 20, 21, 16]. They do not preserve the usual marginal properties, [1, 2, 11], but they do satisfy the generalized forms of the marginals. For example, time marginal in the L-Wigner distribution is the generalized power $|x(t)|^{2L}$ rather than $|x(t)|^2$. Here, we will present the new S-class of distributions which may achieve high concentration at the instantaneous frequency, as high as the distributions from the L-class, while at the same time satisfying energy unbiased condition, time marginal and, for asymptotic signals, frequency marginal.

2. DEFINITIONS

The Scaled variant of the L-Wigner Distribution (S-Distribution) of a signal x(t) is defined by (for more details see Appendix and [54, 34)]:

$$SD_L(t,\omega) = \int_{-\infty}^{\infty} x^{[L]}(t+\frac{\tau}{2L})x^{[L]*}(t-\frac{\tau}{2L})e^{-j\omega\tau}d\tau \tag{1}$$

where $x^{[L]}(t)$ is the modification *of* $x(t)$ obtained by multiplying the phase function by L, while keeping the amplitude unchanged:

$$x^{[L]}(t) = A(t)e^{jL\phi(t)} \tag{2}$$

It is known that, for the ordinary Wigner distribution (eq. (1) with L=1), the Ambiguity function may be defined as its two-dimensional Fourier transform (FT_{2D}), [1, 2]. Here, wi will introduce the S-Ambiguity function and use it to define the S-generalized characteristic function and S-class of distributions.

Definition 1: **The S-Ambiguity function is a FT_{2D} of the S-distribution:**

$$SA_L(\theta,\tau) = \int_u x^{*[L]}(u-\frac{\tau}{2L})x^{[L]}(u+\frac{\tau}{2L})e^{-j\theta u}du. \tag{3}$$

Definition 2: **A product of** $SA_L(\theta,\tau)$ **and an arbitrary function** $_{CL}(\theta,\tau)$ **called the kernel, produces the S-generalized characteristic function:**

$$MG_L(\theta,\tau) = c_L(\theta,\tau)\int_u x^{*[L]}(u-\frac{\tau}{2L})x^{[L]}(u+\frac{\tau}{2L})e^{-j\theta u}du. \tag{4}$$

Definition 3: **The S-class of distributions is an inverse FT_{2D} of the S-generalized characteristic function:**

$$SC_L(t,\omega) = \frac{1}{2\pi}\int_\theta\int_u\int_\tau c_L(\theta,\tau)x^{*[L]}(u-\frac{\tau}{2L})x^{[L]}(u+\frac{\tau}{2L})e^{-j\omega\tau}e^{-j\theta(u-t)}dud\theta d\tau. \tag{5}$$

For L=1 this class of distributions reduces to the Cohen class [1, 6].

Distribution (5) may be understood as an inverse FT_{2D} of the product of $SA_L(\theta,\tau)$ and $c(\theta,\tau)$. Thus, it is equal to the two-dimensional convolution of $\Pi_L(t,\omega) = FT_{2D}\{c_L(\theta,\tau)\}$ and $SD_L(\tau,\omega)$:

$$SC_L(t,\omega) = \frac{1}{2\pi}\int_u\int_v \Pi_L(t-u,\omega-v)SD_L(u,v)dudv \tag{6}$$

All distributions from the S-class may be treated as smoothed versions of the S-distribution.

3. GENERAL PROPERTIES

In this section, we will list some basic properties of the distributions belonging to the S-class. Many of them may be obtained in a straightforward manner, generalizing the ones of the L-class or the Cohen class [1, 2, 38]. These properties will be given without proofs, or any additional explanation. The attention will be paid only to those for which the S-class behaves in a qualitatively different manner than the Cohen class.

1° A distribution from the S-class of distributions is real if its S-generalized autocorrelation function:

$$SRA_L(t,\tau) = \frac{1}{2\pi}\int_\theta MG_L(\theta,\tau)e^{j\theta t}d\theta. \tag{7}$$

is Hermitian, $SRA_L(t,\tau) = SRA^*_L(t,-\tau)$. This condition is satisfied for $c_L(\theta,\tau) = c^*_L(-\theta,-\tau)$

2° The S-class of distributions is time and frequency shift invariant if the kernel $c_L(\theta,\tau)$ is not time (t) and frequency (W) dependent.

3° If a signal is time limited to $|t| < T$ then $SC_L(t,\omega)$ is limited to the same time interval if $C_L(t,\tau) = FT_\theta\{c_L(\theta,\tau)\} = 0$ for $|t/\tau| > 1/(2L)$.

5° If distribution $SC_L(t,\omega)$ corresponds to $x(t)$, then $SC_L(at,\omega/a)$ is the distribution of $|a|^{1/2}x(at)$ provided that $c_L(\theta/a,a\tau) = c_L(\theta,\tau)$.

6° The integral of $SC_L(t,\omega)$ over ω is equal to the signal power $|x(t)|^2$, if $c_L(\theta,0) = 1$:

$$\frac{1}{2\pi}\int_\omega SC_L(t,\omega)d\omega = \frac{1}{2\pi}\int_\omega\int_\tau x^{[L]}(t+\frac{\tau}{2L})x^{[L]*}(t-\frac{\tau}{2L})e^{-j\omega\tau}d\tau d\omega = |x(t)|^2$$

7° If cL(0,0) = 1, then

$$\frac{1}{2\pi}\int_t\int_\omega SC_L(t,\omega)dtd\omega = \int_t |x(t)|^2\,dt = E_x$$

where Ex is the energy of signal x(t).

8° Frequency domain form of SC_L (t,ω) is

$$SC_L(t,\omega) = \frac{L}{4\pi^2}\int_\theta\int_u\int_\tau c_L(\theta,\tau)X_L^*(Lu-\frac{\theta}{2})X_L(Lu+\frac{\theta}{2})e^{j\theta t}e^{-j\tau(\omega-u)}dud\theta d\tau.$$

where $X_L(\omega)$ is the Fourier transform of $x^{[L]}$ (t).

9° If $c_L(0,\tau) = 1$, then the integral of $SC_L(t,\omega)$ over time is

$$\int_t SC_L(t,\omega)dt = \frac{L}{2\pi}\int_\theta\int_u\int_\tau c_L(0,\tau)\,|X_L(Lu)|^2\,e^{-j\tau(\omega-u)}dud\tau = L\,|X_L(L\omega)|^2$$

According to the stationary phase method, [36] , we have

$$X_L(L\omega) = \int_t x^{[L]}(t)e^{-jL\omega t}dt =$$
$$= \int_t A(t)e^{jL\phi(t)}e^{-jL\omega t}dt = A(t_0)e^{jL[\phi(t_0)-\omega t_0]}\sqrt{\frac{2\pi j}{L\phi''(t_0)}}$$

The above relation holds for any signal with continuous A (t) if L → ∞. For asymptotic signal [36] (signal with | A'(t)| << | ϕ'(t)|) for any L. Note that t_0 is a function of ω defined by $\phi(t_0) - \omega = 0$, $\phi''(t_0) \neq 0$. It is easy to conclude that, for asymptotic signals,

$$L\,|X_L(L\omega)|^2 = |X(\omega)|^2$$

meaning that the S-class of distributions, in this case, satisfies the frequency marginal, as well.

4° If an asymptotic signal is band limited to $|\omega| < \omega_m$, then the S-class of distributions is band limited to the same bandwidth if $C_L(\theta,\omega) = FT\tau\{c_L(\theta,\tau)\} = 0$ for $|\omega/\theta| > 1/(2L)$.

10° For the signal $x(t) = A(t)\exp(j\phi(t))$, the mean frequency

$$\langle\omega\rangle_t = \frac{\int_\omega \omega SC_L(t,\omega)d\omega}{\int_\omega SC_L(t,\omega)d\omega}$$

is invariant with respect to L and it is equal to the instantaneous frequency, $\phi'(\tau)$, if $c_L(\theta,0) = 1$ and $\frac{\partial c_L(\theta,\tau)}{\partial\tau}\Big|_{\tau=0} = 0$.

11° Frequency modulated signals representation - The ideal distribution, concentrated along instantaneous frequency, is defined by $2\pi A^2\delta(\omega - \phi'(t))$ or by $A^2W(\omega-\phi'(t))$ if a finite time interval, determined by the window $\omega_L(\tau)=FT^{-1}\{W(\omega)\}$ is used. For the signal $x(t)=Ae^{j\phi(t)}$, this form may be obtained in the Cohen class of distributions, only if the instantaneous frequency is a linear function, $\phi'(t) = at + b$. The distribution which produces this concentration is the Wigner distribution (or pseudo Wigner distribution). If the instantaneous frequency variations are of a higher order than linear, then no distribution (with signal independent kernel) from the Cohen class can produce the ideal concentration.

Theorem 1: **The S-class of distributions for $L\to\infty$ is equal to the ideal form $A^2(t)W(\omega-\phi'(t))$ for any frequency modulated signal $x(t) = A(t)e^{j\phi(t)}$ if the derivatives of the phase function $\phi(t)$ are finite,** A (t) is **continuous, and** $\lim_{L\to\infty} c_L(\theta,\tau) = w(\tau)$, **where $w(\tau)$ is a finite duration window.**

Proof: For a signal of the form $x(t) = A(t)e^{j\phi(t)}$, expanding $\phi(u \pm \tau/2L)$ into a Taylor series around u, up to the third order term, we get:

$$SC_L(t,\omega) = \frac{A^2(t)}{2\pi}\iiint_{-\infty}^{\infty} c_L(\theta,\tau)e^{j\phi'(u)\tau}e^{j\frac{\phi^{(3)}(u+\tau_1)+\phi^{(3)}(u-\tau_2)}{3!L^2}\frac{\tau^3}{8}}e^{j\theta t-j\omega\tau-j\theta u}du\,d\theta\,d\tau \qquad (8)$$

where τ_1 , τ_2 are variables $0 \le |\tau_{1,2}| \le |\tau/2L|$. If $\phi^{(3)}(\tau)$ and $\phi^{(n)}(\tau)$, $n > 3$ are finite and the variable τ may assume only finite values, then for a large L, the value $\lim_{L\to\infty}\exp\left(j\frac{\phi^{(3)}(u+\tau_1)+\phi^{(3)}(u-\tau_2)}{3!L^2}\frac{\tau^3}{8}\right)=1$, and $A(t+t/2L)A(t-t/2L) \cong A^2(\tau)$, so we get:

$$SC_L(t,\omega) \cong \frac{A^2(t)}{2\pi} \iiint_{-\infty}^{\infty} c_L(\theta,\tau) e^{j\phi'(u)\tau} e^{j\theta t - j\omega\tau - j\theta u} du d\theta d\tau \tag{9}$$

This way the S-class of distributions locally linearizes the instantaneous frequency characteristics. Relation (9) may be written in the form:

$$SC_L(t,\omega) \cong A^2(t) \int_{-\infty}^{\infty} \Pi_L(t-u, \omega - \phi'(u)) du \tag{10}$$

where $\Pi_L(t,\omega)$ is the FT_{2D} of $c_L(\theta,\tau)$ and corresponds to the auto-term function in the Cohen class of distributions. If $\lim_{L\to\infty} c_L(\theta,\tau) = \omega(\tau)$, then, for large L, $\Pi_L(t,\omega) = \delta(t)W(\omega)$ and $SC_L(t,\omega) = A^2(t)W(\omega - \phi'(t))$. This form corresponds to the ideal distribution concentration.
Q.E.D.

12° Theorem 2: **For the unity amplitude signals, an** L - *th* **order distribution, belonging to the S-class, may be obtained from its** *L/2 - th* **order form if** $c_L(\theta,\tau) = c_{L/2}(\upsilon,\tau/2)c_{L/2}(\theta-\upsilon,\tau/2)$ **for any** *u*.

Proof: It is evident from (2) that:

$$SA_L(\theta,\tau) = SA_{L/2}(\theta,\tau/2) *_\theta SA_{L/2}(\theta,\tau/2)$$

where $*_\theta$ is a convolution in θ. According to the theorem's assumption, it follows that:

$$MG_L(\theta,\tau) = MG_{L/2}(\theta,\tau/2) *_\theta MG_{L/2}(\theta,\tau/2)$$

Taking the two-dimensional Fourier transform of both sides, we get:

$$SC_L(t,\omega) = \int_\lambda SC_{L/2}(t,\omega+\lambda) SC_{L/2}(t,\omega-\lambda) \frac{d\lambda}{\pi}. \tag{11}$$

Note: The preposition of Theorem 2 is satisfied by the: Wigner, Rihaczek, Page, Levin, . . . type kernels [1, 2]. This theorem will be extensively used for the realization of the distributions belonging to the S-class.

Q. E. D.

Corollary: **For the unity amplitude signals, any** *L-th* **order distribution may be expressed in terms of the** *L/2 - th* **order S-distribution.**

Proof: Relation (11) is valid for the S-distribution. Inserting this relation into (6) we get any distribution expressed in terms of the *L/2 - th* order S-distribution.

Q. E. D.

4. SPECIFIC DISTRIBUTIONS

Here, wi will define some particular distributions belonging to the S-class. Only few interesting properties will be considered for each of them.

4.1 S-distribution

We have already given the definition of the S-distribution, eq. (1), which is the most important member of this class. Since it is taken as a basis for the generalization, obviously its kernel is cL(θ,t) = 1 or for its pseudo form $c_L(\theta,\tau)=\omega_L(\tau)$. The properties and applications of the S-distribution are studied in details in [52, 34]. The realization will be described in the next sections.

4.2 S-Rihaczek distribution

The S-class counterpart of the Rihaczek distribution, in the pseudo form, is defined as:

$$SRD_L(t,\omega) = \int_\tau x^{[L]}(t+\frac{\tau}{L})x^{*[L]}(t)w_L(\tau)e^{-j\omega\tau}d\tau. \qquad (12)$$

This distribution is obtained from the general one with cL(θ,t) = $e^{j\theta\tau/2L}\omega_L(\tau)$

For a frequency modulated signal x(t) = A exp($j\phi$(t)), with $\phi(t) = a + bt + ct^2/2$ after expansion of $\phi(t+\tau/L)$ into a Taylor series, we get:

$$LRD_L(t,\omega) = A^2\delta(\omega-\phi'(t)) *_\omega FT\left\{w_L(\tau)e^{jc\tau^2/(2L)}\right\} \underset{L\to\infty}{\longrightarrow} 2\pi A^2 W(\omega-\phi'(t))$$

This could be expected, since the kernel $c_L(\theta,\tau) = e^{j\theta\tau/2L}\omega L(\tau) \to \omega(\tau)$ as $L\to\infty$ i. e. , its limit satisfies the condition of Theorem 1. But, the convergence in this case is of order 1/L, what is worse than in the S-distribution.

4.3 S-Spectrogram and S-Short time Fourier transform

The S-Spectrogram is defined as the squared modulus of the S-Short time Fourier transform (S-STFT):

$$SSPEC_L(t,\omega) = \left|\int_\tau w_L(\tau)x^{[L]}(t+\frac{\tau}{L})e^{-j\omega\tau}d\tau\right|^2. \qquad (13)$$

Many specific properties of the S-STFT and S-Spectrogram may be easily derived form the widely known properties of the STFT. Here, we will focus the attention only to the one which treats the dependence of frequency and time resolution on the window function. First, assume that the signal x (t) is short, concentrated at t=0 into an interval $\Delta t \rightarrow 0$. If the window $\omega_L(t)$ is time limited to $|t| < T/2$ (where $T >> \Delta t$), then the S-STFT is time limited to $|t| < T/(2L)$ i. e., its duration is $d = T/L$. If we now assume a sinusoidal signal $x(t) = \exp(j\omega_0 t)$ and the same window, we get $SSTFT_L(\omega,\tau) = WL(\omega - \omega_0)e^{jLwot}$. For example, let the window be rectangular. The width of its Fourier transform $W_L(\omega)$ (the width of its main lobe) is $D = 4\pi/L$. The product of the durations d and D (the form of unceraintyprinciple in this case) is $dD = 4\pi/L$ This relation states that the S-STFT, with a given L, can not be localized in time-frequency plane with arbitrary small d and D simultaneously (representing the resolutions in time and frequency directions). But, the previous relation permits an important conclusion: By increasing L, the product dD can be made arbitrary small, meaning arbitrary high resolutions in both directions, simultaneously.

4.4 S-Reduced interference distributions

Although the Wigner distribution satisfies most of the desired properties, it is rarely used in its original form. The main reason lies in the very emphatic cross-term effects. These effects may be even more emphasized in the L-class distribution for $L > 1$, since the L-*th* power of signal may increase the number of cross-terms. Unfortunately, these terms behaves as the regular auto-terms. Thus, the straightforward generalization of the RID distributions (Choi-Williams, Zao-Atlas-Marks, Born-Jordan, Sinc, . . . [1,2]) would reduce only a limited number of cross terms resulting from the product of $x^L(t+\tau/2L)$ and $x^{*L}(t - \tau/2L)$. Originally, the S-class of distributions may be understand as the L-class of the signal $x_M(t) = x^{1/L}(t)$. Thus, even if the amplitude of the original signal $x(t)$ is fast-varying (signal is multicomponent), after the modification $x(t) \rightarrow x^{1/L}(t)$ we get slow-varying amplitude, i. e., a monocomponent signal. Consequently, the recursive method (based on Theorem 2), although very efficient in the realization of the L-class of distributions, will produce qualitatively the same result as the direct realization of the S-class. But, still it is possible to use the S-method in the realization of the cross-term free S-distibutions. The only problem that has to be resolved is how not to increase the order of amplitude during the recursions. Such recursions may be achieved using a slightly modified S-method, for the kernels satisfying the conditions of Theorem 2.

$$SDM_L(t,\omega) = \int_\lambda P(\lambda) SD_{L/2}(t,\omega+\lambda) SD^{(n)}_{L/2}(t,\omega-\lambda)\frac{d\lambda}{\pi}. \quad (14)$$

where subscript $^{(n)}$ denotes a normalized version of distribution $SD_{L}(\tau,\omega)$ i. e. , the distribution $SD_{L/2}(\tau,\omega)$ if all signal components had unity amplitude. Details on the normalization will be provided in the next Section. Thus, starting from the distribution that is cross terms free, we may control (reduce or remove) the cross terms in the subsequent interations using the function $P(\lambda)$ which is of low-pass filter type, while keeping the order of the signal amplitude unchanged. The numerical aspects of the realization of distributions, that may be written in form (14), are also described in [11, 49].

5. ON THE REALIZATION

In our previous work we have described two methods for the L-Wigner distribution realization. They can be directly applied to any distribution from the S-class.

5.1 Direct method

Direct method is based on the modifying signal $x(t)$ into $x^{[L]}(t)$ according to definition, its oversampling L times and keeping unchanged the number of samples used for calculation. Regarding the las assumption, this method is not computationally much more demanding than the realization of any ordinary (L=1) distribution. In the case of multicomponent signals, this method will produce signal power concentrated at the resulting instantaneous frequency, according to Theorem 1.

5.2 Recursive method

Recursive method is based on Theorem 2. This method provided significant advantages in the realization of the L-class distributions: the cross terms are reduced (eliminated); the oversampling is not necessary; computationally, it may be more efficient than the direct method. Particular numerical examples, realized by these methods, along with the details on the methods, may be found in [11, 12, 19, 20, 49, 50]. But, if we want to use this method in the realization of a distribution from S-class wi should modify signal as follows $x(t)$ - $x_M(t) = x^{1/L}(t)$ - $x_M^{[L]}(t)$. Note that in the initial stage, taking the $1/L$ - *th* power of the signal (for large L) we transformed the signal into monocomponent one, what can not be recovered in further steps. Thus, in the case of S-distributions, the recursive method produces qualitatively the same result as the direct method. The only advantage shich remains is that this realization is less noise sensitive than the direct realization. For this reason we propose the scaled recursive method for the realization of a distribution from the S-class.

5.3 Scaled recursive method

Here, we will derive an efficient method for the realization of the S-distribution in the case of a multicomponent signal

$$x(t) = \sum_{i=1}^{P} x_i(t)$$

such that, theoretically its S-distribution is equal to the sum of the S-distributions of each component separately, i. e. :

$$SD_{L,x}(t,\omega) = \sum_{i=1}^{P} SD_{L,x_i}(t,\omega)$$

The marginal properties in this case are:

$$\frac{1}{2\pi}\int_{\omega} SD_{L,x}(t,\omega)d\omega = \sum_{i=1}^{P}|x_i(t)|^2 \quad \text{and} \quad \int_{t} SD_{L,x}(t,\omega)dt = \sum_{i=1}^{P}|X_i(\omega)|^2 \tag{15}$$

Let us start from the Short time Fourier transform of x (t)

$$STFT(t,\omega) = \int_{\tau} w(\tau)x(t+\tau)e^{-j\omega\tau}d\tau = \int_{\tau} w(\tau)A(t+\tau)e^{j\phi(t+\tau)}e^{-j\omega\tau}d\tau \tag{16}$$

As it is known this transform does not have cross-terms between, in time-frequency plane, separated signal components. In order to produce higher order S-distributions we will need an amplitude normalized *STFT* (t,ω) which will be denoted by $STFT^{(n)}(t,\omega)$ and defined as:

$$STFT^{(n)}(t,\omega) = \int_{\tau} w(\tau)e^{j\phi(t+\tau)}e^{-j\omega\tau}d\tau$$

If amplitude A (t) is slow-varying, we may easily get $STFT^{(n)}(t,\omega)$ from $STFT^{(n)}(t,\omega)$ as:

$$STFT^{(n)}(t,\omega) = STFT(t,\omega)\sqrt{\frac{2\pi E_w}{E_x(t)}} \tag{17}$$

where $E_x(t) = \int |STFT(t,\omega)|^2 d\omega$. In the derivation of the above equation the Parseval's theorem is used ($\int |STFT(t,\omega)|_2\, d\omega = \int |\omega(\tau)A(t+\tau)|^2 d\tau$)[1]

If the signal is multicomponent, with slow-varying amplitudes of each component, separated along the frequency axis for any t (i. e. , signal components lie, along ω_i, inside regions Ωi which do not overlap), then:

$$STFT^{(n)}(t,\omega) = \sum_{i=1}^{P} STFT(t,\omega)\sqrt{\frac{2\pi E_w}{E_{x_i}(t)}}\Pi_{\Omega_i}(\omega) \tag{18}$$

where $E_{x'}(t) = \int |STF(t,\omega)|^2\, d\omega$ *and* $\Pi_\Omega(\omega)$ is equal to unity for inside and zero outside (for illustration see Fig. 2). Knowing $STFT(t,\omega)$ and $STFT^{(n)}(t,\omega)$, we may easily realize the distribution:

$$S_1(t,\omega) = \int_\tau w^2(\frac{\tau}{2})A(t+\frac{\tau}{2})e^{j\phi(t+\frac{\tau}{2})}e^{-j\phi(t-\frac{\tau}{2})}e^{-j\omega\tau}d\tau$$

according to the S-method, as:

$$S_1(t,\omega) = \frac{1}{\pi}\int_\theta P(\theta)STFT(t,\omega+\theta)STFT^{*(n)}(t,\omega-\theta)d\theta \tag{19}$$

where $P(\theta)$ is a frequency domain window function, which has to be wide enough to ensure the integration over auto-terms and narrow enough to avoid cross-terms. Recently, we proposed a very simple signal dependent and self adaptive technique, which gives all auto-terms without cross-terms, [35]. After we get cross-terms free $S_1(t,\omega)$, then we may get the S-distribution for L=2, as:

$$SD_2(t,\omega) = \int_\tau w^4(\frac{\tau}{4})A(t+\frac{\tau}{4})A(t-\frac{\tau}{4})e^{j2\phi(t+\frac{\tau}{4})}e^{-j2\phi(t-\frac{\tau}{4})}e^{-j\omega\tau}d\tau$$

convolving two $S_1(t,\omega)$, as

$$SD_2(t,\omega) = \frac{1}{\pi}\int_\theta P(\theta)S_1(t,\omega+\theta)S_1(t,\omega-\theta)d\theta \tag{20}$$

1 Slow-varying amplitude A(t) means that $\omega(t)A(t+\tau) \approx \omega(t)A(t)$. This condition may be written in a less restrictive form. Assume, for example a Hanning window $\omega(\tau)$ and $A(t+\tau) = A(t) + A'(t)\,\tau + A''(t)\tau^2/2$. The scaling factor in (16) remains the same if $A^2(t) >> [A(t) + A(t)A''(t)]\ /6.17 + A''^2(t)/120$ i.e. if A (t), A'(t), A"(t) are of the same order.

where again $P(\theta)$, eliminates (reduces) cross-terms, while the auto-terms are the same as in the original S-distibution of order 2. This procedure may be continued up to any order of the S-distibution. Namely, convolving $SD_2\,(t,\omega)$ and its normalized version $SD_2^{(n)}\,(t,\omega)$ we get $SD4\,(t,)$ and so on. Efficiency of the proposed realization (as well as some other details on the realization itself) will be demonstrated, in the next Section, on a very complex numerical example, including a high amount of noise, [40, 41].

6. EXAMPLES

Example 1: Consider **Gaussian chirp monocomponent signal** of the form:

$$x(t) = Ae^{-at^2/2}e^{jbt^2/2+jct}. \tag{21}$$

a) The **Wigner distribution** of $x(t)$ may be obtained in a closed form as:

$$WD(t,\omega) = \int_{-\infty}^{\infty} x(t+\tfrac{\tau}{2})x^*(t-\tfrac{\tau}{2})e^{-j\omega\tau}d\tau =$$
$$= A^2e^{-at^2}\int_{-\infty}^{\infty} e^{-a\tau^2/4+jbt\tau+jc\tau}e^{-j\omega\tau}d\tau = A^2e^{-at^2}\sqrt{\tfrac{4\pi}{a}}e^{-\frac{(\omega-bt-c)^2}{a}}.$$

This distribution produces the complete concentration at the instantaneous frequency only for $a \to 0$, when the Gaussian chirp signal becomes the purely linear frequency modulated signal, [10, 11, 12]. For any other a, the distribution is spread around the instantaneous frequency.

b) The **S-distribution** of the Gaussian chirp signal is:

$$SD(t,\omega) = \int_{-\infty}^{\infty} x^{[L]}(t+\tfrac{\tau}{2L})x^{*[L]}(t-\tfrac{\tau}{2L})e^{-j\omega\tau}d\tau =$$
$$= LA^2e^{-at^2}\int_{-\infty}^{\infty} e^{-a\tau^2/4+jLbt\tau+jLc\tau}e^{-jL\omega\tau}d\tau = A^2e^{-at^2}\sqrt{\tfrac{4\pi}{a}}Le^{-\frac{(\omega-bt-c)^2}{a/L^2}}.$$

For $a\,/\,L_2 \to 0$ follows:

$$SD(t,\omega) = A^2e^{-at^2}2\pi\delta(\omega-bt-c). \tag{22}$$

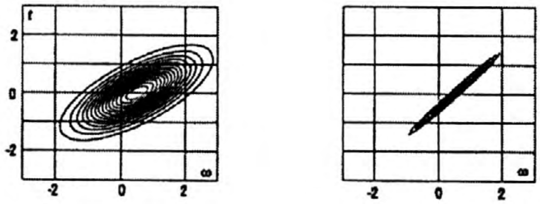

Figure 1: Time-frequency representation of a Gaussian chirp signal: a) Wigner distribution, b) the S-distribution with L=8.

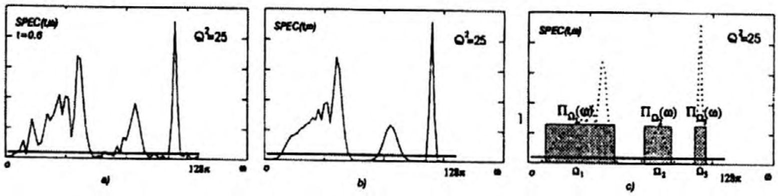

Figure 2: Illustration of the reference level and regions of support: a) Noisy signal, b) Non-noisy signal, c) Regions of support

This is, indeed, just the ideal time-frequency concentration at the instantaneous frequency for any a. The convergence toward the complete concentrated distribution (the distribution of 0 uncertainty) is of order L^2.

The Wigner distribution and the S-distribution of signal (21) are presented in Figs.1a,b, respectively, for $A = 1$, $a = 1$, $b = 1$, $c = 0.5$ and $L = 8$.

Example 2: Consider a multicomponent real signal:

$$x(t) = e^{-4(t-.5)^2}\cos[10\cos(2\pi t) + 30\pi t] + 0.5e^{-4(t-.5)^2}\cos(110\pi t) +$$

$$+0.707e^{-20(t-0.4)^2}\cos(24\pi t^2 + 60\pi t) + n(t)$$

where $n(t)$ is a Gaussian white noise. Signal is sampled within the interval $t \in [0,1]$ at $1/N$, with $N=128$. Hanning window of the unity width, as well as a rectangular window $P\theta$ with signal dependent width, are used. The realization is done according to the procedure described in Subsection 5.3. Here, we will provide some additional details:

1. First, we have to determine regions Ω_i in order to obtain scaling factors in (18). For this purpose we assume the reference level Rlev (t) for a given instant t, as $R_{lev}(t) = \max_{\omega}\{|STFT(t,\omega)|^2\}/Q^2$. The regions Ω_i are

defined by the compact regions where $|STFT(t,w)|^2 \not> R_{lev}(t)$, Fig. 2 a, b, c. Factor Q defines reference level. For non-noisy signals this value may be very high. But, in the noisy cases, in order to avoid false auto-term detection, this factor should not be too large. We found a very appropriate value for non-noisy, as well as for noisy, signals $Q^2=25$. Also, in order to avoid the break of region Ω_i in the cases, as well as in the amplitude of a single auto-term changes sign, we assumed that Ω_i ends not if a single value of $|STFT(t,\omega)|^2$ is bellow $R_{lev}(t)$, but if two subsequent values of $|STFT(t,\omega)|^2$ (at $k\Delta\omega$ and $(k+1)\Delta\omega$) are less than the reference level.

2. After we find regions Ω_i then the scaling factors for each region, according to (18), are determined.

3. Convolution of $STFT(t,\omega)$ and its normalized version $STFT^{(n)}(t,\omega)$ is calculated, according to (19). Here, we used signal dependent rectangular window $P(\theta)$ width. For a given ω inside Ω_i, integration over θ (determined by the width of $P(\theta)$ is performed until any $\omega+\theta$ or $\omega-\theta$ goes outside Ω_i. This way we completely avoid possibility of cross-terms between non-overlapping auto-terms. Also, the accumulation of the noise is kept at the lowest possible level, avoiding all summations outside an auto-term.

4. Finally, convolving two $S_1(t,\omega)$ according to (20), we get cross-term free S-distribution of order 2. Since a high auto-term concentration is achieved in $S_1(t,\omega)$ then a very narrow window $P(\theta)$ in (20) may be used. Even with $P(\theta)=\pi\delta(\theta)$ we get very good results for all considered signals.

5. If one wants to get the S-distribution of a higher order that L=2 (corresponding to the fourthorder distributions) then the steps from 1. to 4. have to be repeated starting from $SD_2(t,\omega)$ instead of $STFT(t,\omega)$, and so on.

On Fig. 3a) Spectrogram is shown. Figs. 3b and 3c presents Wigner distribution, as well as the S-method (auto-terms as in the Wigner distribution, but without cross-terms). Cross-terms free S-distribution, with L=2, is shown in Fig 3d. here, we also presented the marginals obtained from the S-distribution (thick line), as well as the theoretical ones, according to (15), (thin lines). Note that $\omega(\tau)$ does not influence the time-marginal, while the frequency marginal is smoothed by the Fourier transform of the resulting window in the S-distribution.

Case with a high amount of noise (SNR = 4[dB] with respect to the total signal energy or [3B], -3[dB]and [dB] with respect to the first, second and the third signal components, separately, is shown in Fig. 3e and 3f, (S-method and S-distribution of the second order, realized according to the described procedure).

7. CONCLUSION

The S-class of distribution, as a generalization of the S-distribution, is proposed. Method for the efficient realization of the S-class of distributions is presented. Theory is illustrated on the numerical examples. The proposed distributions may achieve arbitrary high concentration at the instantaneous frequency, satisfying the marginal properties. Out of the known distributions, this was only possible in a very special case of the linear frequency modulated signals by the Wigner distribution.

Appendix A
A SHORT REVIEW OF THE WIGNER REPRESENTATION IN THE QUANTUM MECHANICS AND PSEUDO QUANTUM SIGNAL REPRESENTATION

The quantum mechanics form of the Wigner distribution[2], for stationary problems, is given by [3, 4]:

$$W(x,p) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \psi(x+\frac{\hbar\xi}{2})\psi^*(x-\frac{\hbar\xi}{2})e^{-jp\xi}d\xi \tag{23}$$

where ψ is a wave function. The wave function in the previous equation satisfies the Schroedinger equation $-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + j\hbar\frac{\partial\psi}{\partial t} = V(x)\psi$ if $W(x,p,t)$ satisfies the Wigner quantum equation:

$$\frac{\partial W}{\partial t} + \frac{p}{m}\frac{\partial W}{\partial x} = \frac{1}{j\hbar}\left[V(x+\frac{j\hbar}{2}\frac{\partial}{\partial p}) - V(x-\frac{j\hbar}{2}\frac{\partial}{\partial p})\right] W \tag{24}$$

2 Wigner, for his theory, received the Nobel prize in 1963.

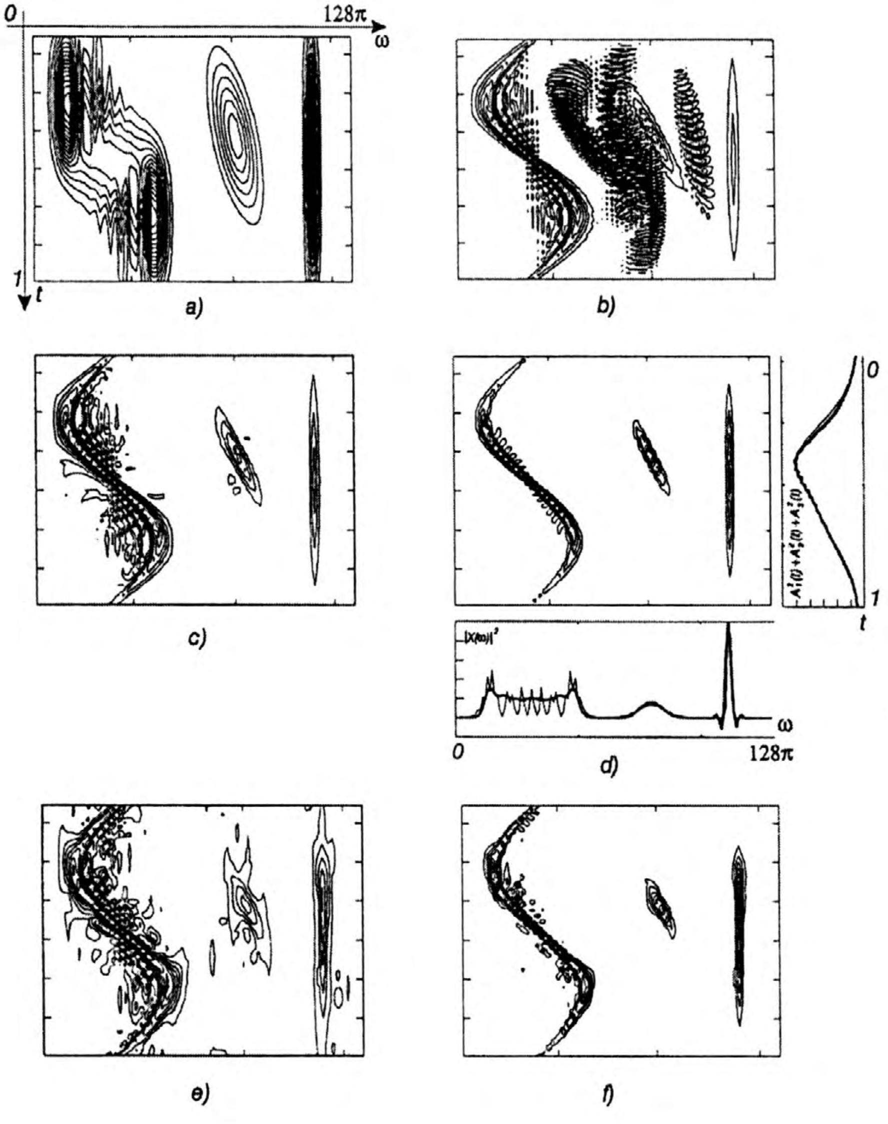

Figure 3: Time-frequency representation of a multicomponent signal: a) Spectrogram, b) Wigner distribution, c) S-method, d) S-distribution with L=2, including marginal properties. e) S-method of noisy signal, f) S-distribution of noisy signal

where h is a constant, $h = h/(2p)$ and $h = 6,6 \times 10^{-34}$ [J s] is the Planck's constant. It may be shown that the Wigner representation and the Schroedinger's one are equivalent, i. e. they uniquely follow from each other [3, 4]. Expression:

may be understood as a quantum correction of the classical Liouville's form, [3, 4]:

$$Q = \frac{1}{j\hbar}\left[V(x + \frac{j\hbar}{2}\frac{\partial}{\partial p}) - V(x - \frac{j\hbar}{2}\frac{\partial}{\partial p})\right] W - V'(x)\frac{\partial W}{\partial p} = -\frac{\hbar^2}{24}V^{(3)}(x)\frac{\partial^3 W}{\partial p^3} + \quad (25)$$

describing the particle motion with: $dx/dt = p/m$ and $dp/dt = -\nabla V(x) = -V'(x)$ where x is the position, m is the mass, $p=mv$ is the momentum of the particle, and $V(x)$ is a potential at the position x. This is a significant prop-

$$\frac{\partial W}{\partial t} + \frac{p}{m}\frac{\partial W}{\partial x} = V'(x)\frac{\partial W}{\partial p}. \quad (26)$$

erty of the Wignei representation, since it may be used to transform the solutions from the classical to the quantum forms [4, 22] or to deal with problems with mixed (quantum and classical - semi classical) variables.

Any function

$$\psi(x) = A(x)e^{j\frac{1}{\hbar}\phi(x)} = \psi_0^{[1/\hbar]}(x) \quad (27)$$

with $\psi_0(x) = A(x)e^{j\phi(x)}$being h-independent, is the solution of the Schroedinger's equation if $A(x) = A$ and $\phi(x) = ax + b$ or if:

$$\left|\frac{A''(x)}{A(x)}\hbar + j\left[\phi''(x) + \phi'(x)\frac{A'(x)}{A(x)} + \phi(x)\frac{A'(x)}{A(x)} + \right]\right|\hbar \ll |\phi'(x)|^2 \quad (28)$$

when:

$$[\phi'(x)]^2 = 2mV(x). \quad (29)$$

In the light of (28), we mention again that h is of order 10^{-34}. Thus, for any function (27), satisfying (28), the Wigner distribution may be written in form:

$$W(x,p) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \psi_0^{[1/\hbar]}(x + \frac{\hbar\xi}{2})\psi_0^{*[1/\hbar]}(x - \frac{\hbar\xi}{2})e^{-jp\xi}d\xi \quad (30)$$

The wave function defined by (27), along with (28), (29) and with $A(x) = A$, is the form of solution for the Schroedinger's equation, proposed by Wentzel [27]. It is efficiently used in the quantum mechanics problems, especially for the transmission coefficients calculations. Formally, the same form as (27), with $A(x) = A$, is used as a wave function in the Feynman's theory of path integrals [28]:

$$\psi[x(t)] = Ae^{j\frac{1}{\hbar}S[x(t)]}$$

where $S[x(t)] = \int_{t1}^{t2} L(\dot{x},x,t)dt$ is h-independent and $L = m\dot{x}^2/2 - V(x,t)$ is the Lagrange's operator.

Here, we will also indicate that the uncertainty principle in the Wigner representation, in quantum mechanics, 4 states that the product of durations in the directions of x and p axis in the (x,p) plane can not be arbitrary small, i. e. , the wave function can not be concentrated simultaneously in $(\Delta x, \Delta p)$ in an arbitrary small interval. **This product is greater or equal to** $h^2/4$

In the signal analysis, the variables: frequency ω and time t are used, instead of p and x. The Wigner distribution with these coordinates is derived as:

$$WD(t,\omega) = \int_{-\infty}^{\infty} s(t+\frac{\tau}{2})s^*(t-\frac{\tau}{2})e^{-j\omega\tau}d\tau. \quad (31)$$

The equivalence between the quantum mechanics definition (23) and signal analysis definition (31) is obvious with: $\psi(x)$ - $s(t)$ To be precise, the analogy exists between the spaces (ω,t) and (k, x), rather than between (ω,t) and (p, x). Equation (31) follows from (23) with $k = p/\hbar \rightarrow \omega$ and $x \rightarrow t$ (the presence of factor $1/2\pi$ is due to the different forms of the Fourier transforms commonly used in the quantum mechanics and signal analysis and we intentionally did not want to modify any of them), [6, 29, 30, 31, 32, 33]. Of course, in the signal analysis **a signal need not to satisfy the Schroedinger's equation** (like the wave function in (23) does), but it is rather obtained as a result of some physical process or theoretical analysis.

For the signal $s(t) = A\exp(j\phi(t))$ the Wigner distribution (31) assumes the form:

$$WD(t,\omega) = A^2 \int_{-\infty}^{\infty} e^{j[\phi(t+\frac{\tau}{2})-\phi(t-\frac{\tau}{2})]}e^{-j\omega\tau}d\tau =$$

$$= A^2 \int_{-\infty}^{\infty} e^{j[\phi(t+\frac{\tau}{2})-\phi(t-\frac{\tau}{2})]-j\phi'(t)\tau}e^{j\phi'(t)\tau}e^{-j\omega\tau}d\tau.$$

Note that the factor $A^2 \int_{-\infty}^{\infty} e^{j\phi'(t)\tau}e^{-j\omega\tau}d\tau$ produces the ideal distribution concentration $2\pi A^2\delta(\omega-\phi'(t))$, while the term with the phase, which is formally similar to (25).

$$Q = j\left[\phi(t+\frac{\tau}{2}) - \phi(t-\frac{\tau}{2})\right] - j\phi'(t)\tau = j\frac{1}{24}\phi^{(3)}(t)\tau^3 + \ldots \quad (32)$$

produces the spread of distribution around the instantaneous frequency.

Factor Q is equal to zero if the instantaneous frequency $\phi'(t)$ is linear function of time,i. e. , if $\phi^{(n)}(t) \equiv 0$, for $n \geq 3$. In the quantum mechanics, the quantum correction term Q was equal to zero for the potential function such that the terms with $\hbar^{n-1}$, $V^{(n)}(x)$, $n \geq 3$are negligible. This is in a complete agreement with (29), where the linear function $\phi'(t)$ corresponds to the quadratic function $V(x)$.

We now pose the question: Is it possible to use a form of the Wigner distribution in the signal analysis other than (31)? In particular, we look after a form which would keep a constant corresponding to in (25) and (23). This would be of great help in the analysis of the non-linear frequency modulated signals, especially since in this analysis one is not restricted to the physical (real world) value of this constant. Thus, we would have an opportunity to choose its most sutiable value. Now, we will present a reflection which led to that form of the Wigner distribution.

Let us for the sake of argument, transcend the real world and enter the realm of a thought experiment. Assume that there are fictitiou „worlds" in which may assume some other constant values, not just the conventional one. This fictitious constants will be denoted by Forms associated with this new constant will be, in the sequel, referred to as **"pseudo quantum forms"**.

Having in mind this freedom, we may reinterpret the above signal processing definitions in the following way: On the basis of the **signal**, given in the signal analysis, we generate the „pseudo wave function" with the corresponding „**pseudo particle**" having the „**pseudo-momentum**" $\varphi=\hbar_f\omega$The transformation of a signal into the „**pseudo wave function**" is done according to (27). Thus, the signal analysis form of the Wigner distribution (31) may be treated as a special case of the „pseudo quantum" form of (23) with $\hbar_f = 1$ (in the „world" where $\varphi \equiv \omega$). Now, we may pose the question: Why to be restricted to $\hbar_f = 1$? or: Is it possible to obtain any improvement in the signal analysis using some other values for $\hbar_f$?

It is obvious from the quantum mechanics forms that the uncertainty may be decreased by using smaller values of $\hbar_f$. This means, if we are able, for a given signal, to form a „pseudo wave function" having different „pseudo momentums" in different fictitious „worlds" (with different constants $\hbar_f$), then we could always go to a „world" with a small uncertainty and analyze the signal in that „world" (in its (φ,t) plane). For example, if the signal is linear frequency modulated, then the Wigner distribution in the „world" $\hbar_f = 1$produces the ideal concentration of the signal energy at its instantaneous frequency, eq. (32). So, in this case there is no need to go to any „world" with smaller $\hbar_f$. But, if the signal is not linear frequency modulated, we should go to smaller in order to improve concentration. How far to go with decreasing depends on how significantly the non-linearities are exhibited in the signal (i. e. , how large is the influence of the higher order terms).

In this way, **by varying he value of $\hbar_f$ we are in position to influence the uncertainty of the „pseudo quantum" (time - „pseudo momentum") signal presentation, while, as it is shown in the paper, keeping the most important properties of the time-frequency representation invariant. Abstracting the physical sense of the quantum mechanics representation we defined the „pseudo quantum" signal representation. Its basic form, according to the above consideration, with:**$\hbar_f = 1/L$, $j \to x$, $\dot{x} \to t$, $p \to \omega$, is the S-distribution:

$$SD(t,\omega) = \int_{-\infty}^{\infty} x^{[L]}(t+\frac{\tau}{2L})x^{*[L]}(t-\frac{\tau}{2L})e^{-j\omega\tau}d\tau \tag{33}$$

with the spread factor:

$$Q = jL\left[\phi(t+\frac{\tau}{2L}) - \phi(t-\frac{\tau}{2L})\right] - j\phi'(t)\tau = j\frac{1}{24L^2}\phi^{(3)}(t)\tau^3 + \ldots \tag{34}$$

APPENDIX B

S-WAVELET DISTRIBUTIONS

Expressions (5) and (6), as well as the complete theory presented in the paper, may be easily extended to the time-scale energy distributions [17]:

$$SC_L(t,a) = \frac{1}{2\pi}\int_u\int_v \Pi_L(\frac{t-u}{a}, \omega_0 - av)SD_L(u,v)dudv \tag{35}$$

where a is a scale factor $\alpha \equiv \omega_0/\omega$. The properties and special cases of this class of distributions may be derived starting from the previous ones and the ones described in 11, 17.

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