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CHARACTERISTIC CLASSES OF A HYPERCOMPLEX VECTOR BUNDLES

A b s t r a c t

We consider a hypercomplex vector bundles over a Riemannian manifolds. We study their first and second Chern classes and relations between them. We obtain the vanishing of the first Chern class and the estimations of the second Chern numbers, in particular when the bundle allows formally holomorphic or canonical holomorphic connection.

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Key words: hypercomplex vector bundle, formally holomorphic connection, canonical holomorphic connection

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KARAKTERISTIČNE KLASE HIPER-KOMPLEKSNIH VEKTORSKIH RASLOJENJA

Izvod

U radu se razmatraju hiperkompleksna vektorska raslojenja Rimanovih mnogostrukosti. Izučavaju se Černove klase (prva i druga) i odnosi između njih. U slučaju kada raslojenje dopušta, formalno holomorfnu ili kanonsku holomorfnu povezanost, dobija se anuliranje prve Černove klase i ocjene za druge Černove brojeve.

1. INTRODUCTION

Let ξ be an almost hyprecomplex vector bundle over a Riemannian manifold M. If this is the case, we have a family of induced complex vector bundles and we may than ask what relations exist between Chern classes and Chern numbers of such bundles.
In ([1]) are studied hyper-Kähler and quaternionic-Kähler manifolds, i.e. Riemannian manifolds admitting parallel almost hypercomplex and parallel quaternionic structures. Also, Piccinni ([5]) studied canonical line bundle of the quaternionic projective space which apears as non-tangent bundle admitting almost hypercomplex structure.
In this paper we generalize these notions for the case of an arbitrary almost hypercomplex bundle and consider Chern classes and Chern numbers of induced complex vector bundles.
In Section 2 we introduce the notion of a hypercomplex vector bundles as generalization of tangent bundles of hyper-Kähler manifolds. We also generalize a notion of formally holomorphic connection and introduce canonical holomorphic connection. Particulary, the Riemannian connection of a hyper-Kähler manifolds is formally holomorphic and the metric connection of the canonical line bundle of the quaternionic projective space is canonical holomorphic.
In ([1]) is obtained that the first Chern class of induced complex vector bundle from a tangent bundle of a hyper-Kähler manifold vanishes. So, in theorem 1 we prove that the same states for hypercomplex vector bundles.
In theorem 2 we prove that the second Chern classes are equal for all

induced complex vector bundles.
We also use formally holomorphic and canonical holomorphic connection and following ([3]) we get estimations for the second Chern numbers.
In section 5 we cover all introduced notions with examples.

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2. HYPERCOMPLEX VECTOR BUNDLES

Let ξ be a real vector bundle over a manifold M, where M has real dimension n, and each fiber of ξ has real dimension $4k$. The bundle ξ is called *almost hypercomplex* if there are two almost complex structures I and J satisfying $IJ = -JI$. When this is the case, we can define another almost complex structure K by $K = IJ$. These three almost complex structures then satisfy quaternionic relations, $IJ = -JI$, $IK = -KI$, $JK = -KJ$, $I^2 = J^2 = K^2 = -1$. The triple $H = (I, J, K)$ is called *almost hypercomplex strusture* on ξ.

Definition 1 *A connection ∇ on an almost hypercomplex bundle ξ over a manifold M will be called* hypercomplex *provided*

$$\nabla(I) = \nabla(J) = \nabla(K) = 0. \tag{1}$$

Definition 2 *An almost hypercomplex vector bundle will be called* hypercomplex *if there is connection ∇ provided*

(i) ∇ is hypercomplex,

(ii) ∇ preserves the fibre metric h, i.e.

$$Uh(s,t) = h(\nabla_U s, t) + h(s, \nabla_U t), \; U \in (M), \; s, t \in S(\xi).$$

Remark. It is well known that a tangent bundle of hyper-Kähler manifold ([1]) is hypercomplex vector bundle. The canonical line bundle of the quaternionic projective space is the main example of the hypercomplex vector bundle not appearing as a tangent bundle of a hyper-Kähler manifold.

A manifold M is called *locally hypercomplex* if there is covering $U = \{U_i\}$ of M where U_i are open sets, and on each of them almost hypercomplex structure, such that for every $x \in U_i \cap U_j$, the subspace of space of endomorphisms $End(T_x M)$ is the same for i and j.

A *holomorphic* subspace of a real tangent space of locally hypercomplex manifold is a subspace invariant under given quaternionic structure.

Definition 3 *A connection ∇ on an almost hypercomplex vector bundle ξ over locally hypercomplex manifold M will be called a* formally holomorphic *provided*

(i) the bundle ξ is hypercomplex,

(ii) the curvature tensor R of ∇ satisfies

$$R_{UT} = R_{IUIT} = R_{JUJT} \quad \textit{for} \quad U, T \in (M). \tag{2}$$

Remark. It is well known that the Levi-Civita connection of a hyper-Kähler manifold is formally holomorphic. Moreover, this type of connection is considered in the case of almost complex vector bundles ([3]), such it is naturally to consider it here.

Definition 4 *A connection ∇ on an almost hypercomplex vector bundle ξ over locally hypercomplex manifold M will be called* canonical holomorphic *provided:*

(i) the bundle ξ is hypercomplex,

(ii) the curvature tensor R of ∇ satisfies

$$R_{UIU} = R_{JUIJU}, \tag{3}$$

$$R_{UJU} = R_{KUJKU}, \quad \textit{for} \quad U \in (M).$$

Remark. It has been shown that the metric connection of the canonical line bundle of the quaternionic projective space is not formally holomorphic, but we have symmetry properties the same as in (3). It was the motivation for introducing the notion of the canonical holomorphic connection.

3. THE FIRST CHERN CLASS

Let ξ be a quaternionic vector bundle over a manifold M, where M has dimension n and each fiber of ξ has real dimension 4. Let e, Ie, Je, Ke be a locally defined real ortonormal base for ξ.

The hypercomplex connection $\bigtriangledown$ on ξ with respect to this base is given by the matrix ω 1-forms such that

Let be

$$\bigtriangledown(e) = \omega_1 e + \omega_2 Ie + \omega_3 Je + \omega_4 Ke. \tag{4}$$

Since $\bigtriangledown(Ie) = I(\bigtriangledown(e))$, we get

$$\bigtriangledown(Ie) = -\omega_2 e + \omega_1 Ie - \omega_4 Je + \omega_3 Ke. \tag{5}$$

Similar, relations hold for J and K.

By writing

$$iX = IX \text{ for } X \in \xi_x, \ x \in M,$$

we can view (ξ, I) as a complex vector bundle.

Local unitary frame fields are given by e, Je or e, Ke. With respest to base e, Je the connection matrix of $\bigtriangledown$ (which is now complex connection on the bundle (ξ, I)) is given by

$$\omega_{11} = \omega_1 + i\omega_2, \ \omega_{12} = \omega_3 + i\omega_4, \tag{6}$$

$$\omega_{21} = -\omega_3 + i\omega_4, \ \omega_{22} = \omega_1 - i\omega_2.$$

The first Chern form is given by ([2], [4])

$$\gamma_1 = \frac{1}{2\pi i}(\Omega_{11} + \Omega_{22}) = \frac{1}{2\pi i} d\omega_1, \tag{7}$$

so we get $c_1(\xi, I) = 0$, since the $c_1(\xi, I)$ is real cohomology class.

The previous can be expressed in terms of curvature tensor R of the connection $\bigtriangledown$ on ξ. If $f_1, ..., f_n$ is a local frame field for the tangent bundle of a manifold M, we write

$$R_{f_i f_j}(s) = R_{ij}(s) \text{ for } s \in S(\xi).$$

Let be

$$R_{ij}(e) = R_{ij1}{}^1 e + R_{ij1}{}^2 Ie + R_{ij1}{}^3 Je + R_{ij1}{}^4 Ke. \tag{8}$$

The curvature tensor of the bundle (ξ, I) is given by

$$S_{ij1}{}^1 = R_{ij1}{}^1 + iR_{ij1}{}^2,\ S_{ij1}{}^2 = R_{ij1}{}^3 + iR_{ij1}{}^4, \tag{9}$$

$$S_{ij2}{}^1 = -R_{ij1}{}^3 + iR_{ij1}{}^4,\ S_{ij2}{}^2 = R_{ij1}{}^1 - iR_{ij1}{}^2.$$

So, in this case we get ([2], [4])

$$\gamma_1 = \frac{1}{2\pi i} \sum_{i,j=1}^{n} R_{ij1}{}^1 dx_i \wedge dx_j. \tag{10}$$

Since $R_{ij1}{}^1 = 0$, we get that $\gamma_1 = 0$, i.e., the first Chern class

$$c_1(\xi, I) = 0. \tag{11}$$

We can consider almost complex structure

$$L_z = aI + bJ + cK, \quad \text{where} \quad z = (a, b, c) \in S^2.$$

Moreover, there are always almost complex structures L_u and L_v provided

$$L_z L_u = -L_u L_z \quad \text{and} \quad L_v = L_z L_u.$$

Namely, $L_u = a_1 I + b_1 J + c_1 K$, where $u = (a_1, b_1, c_1) \in S^2$ and the condition $L_z L_u = -L_u L_z$ implies $aa_1 + bb_1 + cc_1 = 0$, i.e., the point (a_1, b_1, c_1) belongs to the circle

$$\begin{cases} aa_1 + bb_1 + cc_1 = 0 \\ a_1^2 + b_1^2 + c_1^2 = 1. \end{cases} \tag{12}$$

We write $L_v = L_z L_u$. Now, we have previous situation, three almost complex structures L_z, L_u, L_v that are related to each other by quaternionic relations. According to this fact, every result for a complex bundle (ξ, I) which depends only on the existence of the structures J and K is stated for an arbitrary induced complex bundle (ξ, L_z).

We obtain the same result if we consider the hypercomplex bundle ξ of an arbitrary range $4k$ over manifold M of an arbitrary dimension n.

Therefore:

Theorem 1 *The first Chern class of an arbitrary complex bundle* (ξ, L_z), $z \in S^2$, *which is induced by hypercomplex bundle* ξ *over a Riemannian manifold vanishes.*

Remark. It is well known that the first Chern class of a complex bundle which is induced by tangent bundle of hyper-Kähler manifold vanishes ([1]).

4. THE SECOND CHERN CLASS AND THE SECOND CHERN NUMBER

We will consider only a real bundles of range four.

Let first consider the relation between the second Chern classes of the bundles (ξ, L_z) that are induced by hypercomplex bundle ξ in the case when the manifold has dimension 4. For the bundle (ξ, I) we get that the second Chern form is given by

$$\gamma_2 = -\frac{1}{2\pi^2}(h(R_{12}(e), R_{34}(e)) - h(R_{13}(e), R_{24}(e)) + h(R_{14}(e), R_{23}(e))). \quad (13)$$

We will get the same result if we compute the second Chern forms of the bundles (ξ, J) and (ξ, K). The previous result doesn't depend of the dimension of manifold M:

Theorem 2 *The second Chern classes of all complex bundles* (ξ, L_z), $z \in S^2$, *that are induced by hypercomplex vector bundle* ξ, *are equal.*

Moreover, it can be noticed that even the second Chern forms are equal.

Now, we pose the question:

What can be said about the nonnegativity or nonpositivity of the Chern numbers of the complex bundles induced by hypercomplex bundles?

Precisely, we have the following:

Theorem 3 *Let* M *be a compact locally hypercomplex manifold of dimension 4 and let* (ξ, L_z), $z \in S^2$, *be a complex vector bundle induced by hypercomplex bundle* ξ. *Let* ∇ *be a formally (canonical) holomorphic connection on* ξ. *Then the second Chern number is nonnegative (nonpositive) and equality holds if and only if the existing formally (canonical) holomorphic connection is flat.*

Proof : If we consider a Riemannian manifolds of dimension 4, than in the case of hypercomplex bundle of range 4, we get

$$\gamma_2 = -\frac{1}{2\pi^2}(h(R_{12}(e),R_{34}(e)) - h(R_{13}(e),R_{24}(e)) + h(R_{14}(e),R_{23}(e))). \quad (14)$$

If the existing connection is formally (canonical) holomorphic then

$$R_{12} = -R_{34}, \quad R_{13} = R_{24}, \quad R_{14} = -R_{23}, \quad (15)$$

$$(R_{12} = R_{34}, \quad R_{13} = -R_{24}, \quad R_{14} = R_{23}),$$

so,

$$\gamma_2 = \frac{1}{2\pi^2}(\|R_{12}(e)\|^2 + \|R_{13}(e)\|^2 + \|R_{14}(e)\|^2) dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4 \quad (16)$$

$$(\gamma_2 = -\frac{1}{2\pi^2}(\|R_{12}(e)\|^2 + \|R_{13}(e)\|^2 + \|R_{14}(e)\|^2) dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4).$$

Thus, the second Chern number

$$c_2[(\xi, L_z)] = \int_M \gamma_2 \geq 0 \ \ (\leq 0). \quad (17)$$

It is obvious that $c_2[(\xi, L_z)] = 0$ if and only if the existing connection is flat. □

So, the second Chern number of a tangent bundle of a four-dimensional hyper- Kähler manifold is nonnegative and equality holds if and only if the Riemannian connection is flat. For example, K^3 surface is hyper-Kähler, it has Ricci flat metric which is not flat and it's Euler characteristic is

$$\chi(K^3) = c_2[K^3] = 24.$$

Also the second Chern number of the canonical line bundle of the quternionic projective space is nonpositive.

The previous result can be generalized:

Theorem 4 *Let (ξ, L_z) $z \in S^2$, be a complex vector bundle induced by hypercomplex vector bundle which allows a formally holomorphic connection. Then the second Chern form γ_2 is nonegative on 4-dimensional holomorphic subspaces of tangent space to M.*

Proof : On four-dimensional formally holomorphic subspaces of tangent space to M, we have locally base

$$f_1, \quad f_2 = I f_1, \quad f_3 = J f_1, \quad f_4 = K f_1$$

so we get

$$\gamma_2 = -\frac{1}{2\pi^2}(h(R_{12}(e), R_{34}(e)) - h(R_{13}(e), R_{24}(e)) + h(R_{14}(e), R_{23}(e))). \quad (18)$$

If the connection is formally holomorphic then

$$R_{ij} = R_{IiIj} = R_{JiJj} = R_{KiKj}, \quad (19)$$

thus

$$\gamma_2 = \frac{1}{2\pi^2}(\|R_{12}(e)\|^2 + \|R_{13}(e)\|^2 + \|R_{14}(e)\|^2) dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4. \quad (20)$$

So, we get that $\gamma_2 \geq 0$ and equality holds if and only if $R = 0$. □

Theorem 5 *Let (ξ, L_z), $z \in S^2$, be a complex vector bundle that are induced by hypercomplex bundle ξ which allows the canonical holomorphic connection. Then the second Chern form γ_2 is nonpositive on 4-dimensional holomorphic subspaces of tangent space to M.*

Remark. In ([3]) is showed that the second Chern form of a complex vector bundle over an almost complex manifold, with the presence of a formally holomorphic connection, is nonnegative on 4-dimensional holomorphic subspaces of a tangent space to M, if the holomorphic bisectional curvature of this bundle is nonnegative.

5. EXAMPLES

The main example in this paper is the canonical line bundle of the quaternionic projective space, see ([5]). Namely, it is the main example of the hypercomplex vector bundle not apearing as a tangent bundle of the hyper-Kähler manifold.

Let $\mathbb{H}^n$ be a left quaternionic vector space of n-tuples $(h_1, h_2, ..., h_n)$ of quaternions.

Let $\gamma^1(\mathbb{H})$ be the left quaternionic line bundle, i.e., the fibres are the lines through the origin in $\mathbb{H}^2$, projecting in the points of the quaternionic projective line $\mathbb{H}P^1$. The two open subsets $U_\tau = \{[(h_0, h_1)] \in \mathbb{H}P^1; h_\tau \neq 0\}$, $(\tau = 0, 1)$ trivialize $\gamma^1(\mathbb{H})$.

Let $H^* = \mathbb{H} - \{0\}$. We call linear section with coefficient $q \in H^*$ of $\gamma^1(\mathbb{H})$ over U_0 (resp. U_1) any nowhere zero section that corresponds to the section $[(1, h_1)] \in U_0 \to (q, qh_1) \in \mathbb{H}^2 - \{0\}$ (resp. $[(h_0, 1)] \in U_1 \to (qh_0, q) \in \mathbb{H}^2 - \{0\}$).

The standard hermitian product in $\mathbb{H}^2$ defines on $\gamma^1(\mathbb{H})$ hermitian metric

$$H = \gamma \circ s \tag{21}$$

where s is a local nowhere zero section of $\gamma^1(\mathbb{H})$ and γ is an hermitian form in $\mathbb{H}^2$ defined by $\gamma = h_0\overline{h_0} + h_1\overline{h_1}$. Piccinni ([5]) proved the following lemma:

Lemma 1 *In $\gamma^1(\mathbb{H})$ there exist a unique connection compatibile with the metric defined by γ and satisfying the following condition. For any linear section with real coefficient over U_0 or U_1 the associated connection form is a quaternionic (1,0)-form in the non-homogenus coordinates h or h^{-1}, where $h = h_0^{-1}h_1$.*

It is obvious that we have the hypercomplex structure on this bundle. The linear section with coefficient 1 will be a base over U_0 i.e. $e = (1, h)$. According to previous lemma we write

$$\nabla(e) = \omega_0 e = dh\overline{h}(1 + h\overline{h})^{-1}e, \tag{22}$$

i.e.

$$\nabla(e) = \omega^0 e + \omega^1 Ie + \omega^2 Je + \omega^3 Ke.$$

The components of the curvature tensor R are given by

$$R_{12} = R_{34} = -\frac{2}{(1 + x_1^2 + x_2^2 + x_3^2 + x_4^2)^2} Ie, \tag{23}$$

$$R_{13} = -R_{24} = \frac{2}{(1 + x_1^2 + x_2^2 + x_3^2 + x_4^2)^2} Je,$$

$$R_{14} = R_{23} = -\frac{2}{(1 + x_1^2 + x_2^2 + x_3^2 + x_4^2)^2} Ke.$$

It is obvious that the first Chern class is zero, where for the second Chern form we get

$$\gamma_2 = -\frac{1}{\pi^2(1+x_1^2+x_2^2+x_3^2+x_4^2)^4} dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4. \tag{24}$$

Therefore, for the second Chern number we obtain

$$c_2 = \int_{HP_1} \gamma_2 \leq 0, \tag{25}$$

i.e.,

$$c_2 = -\frac{1}{6}. \tag{26}$$

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