M. Jaćimović, I. Krnić, O. Obradović
Department of Mathematics, University of Montenegro, Podgorica, Montenegro

Wellposedness and regularization of minimizing sequences in quadratic programming problems in Hilbert space

Abstract

Wellposdness of a minimization problem means that small perturbations of initial data of a problem do not produce big changes in its set of solutions. One of the important moments in the examination of wellposedness of a minimization problem is the behavior of its minimizing sequences. In this paper we present some necessary and sufficient conditions for convergence of the minimizing sequence of quadratic minimization problem with linear and quadratic constraints. For one class of these problems we also present a method of stabilization which produces convergent minimizing sequences.

Key words: quadratic programming, stability, well-posedness, minimizing sequences

1. Introduction

An optimization problem is said to be wellpossed if its small perturbations do not produce big changes in its set of solutions. This statement allows different formalizations. In this paper we will deal with Tikhonov's concept of wellposedness which is related to the minimization problem

$$J(u) = \|Au - f\|^2 \to \inf, \; u \in U, \tag{1}$$

where $U \subseteq H$ is a closed convex set in a Hilbert space H, $A : H \mapsto F$ is a linear bounded operator from H to Hilbert space F, and $f \in F$ is a fixed element. In what follows the set U will be defined by linear and quadratic constraints:

$$U = \{u \in H : \|Bu\|^2 \leq r^2, \; \langle c, u\rangle \leq \beta\}, \tag{2}$$

where $B : H \to G$ is a bounded linear operator form H to Hilbert space G and $c \in H$.

The problem of minimization of the functions

$$J_1(u) = \|x(T,u) - z\|^2_{R^n}, J_2(u) = \|x(t,u) - z(t)\|^2_{L_2} = \int_0^T \|x(t,u) - f(t)\|^2 dt,$$

where $z \in R^n$, $f \in L_2[0,T]$, and $x = x(t,u)$ is a solution of the system of differential equation

$$x'(t) = B(t)x(t) + D(t)u(t),\ t \in (0,T), x(0) = 0 \in R^n,$$

with given matrices $B(\cdot) = (b_{ij}(\cdot))_{n\times n}$ and $D(\cdot) = (d_{ij}(\cdot))_{n\times r},\ a_{ij})\cdot), b_{ij}(\cdot) \in L_\infty[0,T]$, are the examples of this type of problem. These conditions guarantee the existence of the solution $x(t,u) \in H^1_n[0,T]$ of the previous system for each $u \in L^r_2[0,T]$. In these examples, the sets of minimization are usually defined by one or two constraints of the type

$$\|b(t)u(t)\|^2_{L^r_2} = \int_0^T |b(t)u(t)|^2 dt \le r^2,\ \|x(t,u)\|^2 = \int_0^T |x(t,u(t))|^2 d\,t \le r^2,$$

$$\|x(T,u)\|^2 \le r^2, \int_0^T c(t)x(t) \le \beta, \langle c, x(T,u)\rangle_{R^n} \le \beta.$$

Problem (1) is is said to be wellpossed according to Tikhonov if every minimizing sequence (u_n) *of this problem converges to the nonempty set of solutions.*

Note that a sequence $(u_n), \{u_n : n \in \mathbb{N}\} \subseteq U$ *is minimizing for the problem* $J(u) \to \inf,\ u \in U$ *if* $J(u_n) \to \inf\{J(u) : u \in U\}$.

The minimization problem of strongly convex function on a closed convex subset of a Hilbert space is (Tikhonov's) well-posed. Function J in (1) is convex but it is not necessarily strongly convex and both the existence of the solution and behavior of the minimizing sequences of this problem depend on the properties of the operator A and set U. Let us note that, in optimal control problems, the set U is often given by one or two linear or quadratic constraints.

In this paper, we will investigate the minimizing sequences of this problem, assuming that the set U is given by linear and quadratic constraints. We will also present some classes of regularization methods for construction of convergent sequences in case when the set U is a half-space of real Hilbert space and prove the convergence of the regularized solution. Let us remark that in optimal control problem the set U is usually given by one or two linear and quadratic constraints.

Let us emphasize that all our results related to well-posedness were obtained under the assumption that all the initial data are known exactly; well-posedness related to inexact initial data will not be considered here.

With the aim that of presenting the complete results in this paper we have also included the part of the results published in [5] and [6].

2. Preliminaries

Let us introduce the following notation: $A : H \mapsto F$ and $B : H \mapsto G$ – bounded linear operators from Hilbert space H to Hilbert spaces F and G; $Cu = \langle c, u\rangle$, $c \in H$ –the linear functional on H; $\mathcal{L}(\mathcal{M})$ – the linear hull of the set $M \subseteq H$; I–the identity operator; $R(A)$ – the range of the operator A; $A(U) = \{Au : u \in U\}$; $Ker\, A$ – the null-space of A; $\overline{M}$–the closure of the set $M \subseteq H$; $L^{\perp}$– the orthogonal complement of the subspace L; P– the orthogonal projection operator from H to $\overline{R(A^*)}$; Q– the orthogonal projection operator from H to $\overline{R(B^*)}$; P_r– the orthogonal projection operator from F to the closed and convex set $\overline{A(U)}$; B_1 – the restrictions of the operators B to the subspace $Ker\, A \cap Ker\, C$ and A_1 – the restriction of the operator A to the subspace $Ker\, B \cap Ker\, C$; A_B – the restriction of the operator A to the subspace $Ker\, B$ and $S = \{u \in H : \|Bu\| = r, \langle c, u\rangle = \beta\}$ – the intersection of the boundaries of the elipsoid $U_1 = \{u \in H : \|Bu\| = r\}$ and half-space $U_2 = \{u \in H : \langle c, u\rangle = \beta$.

The operator A produces the following orthogonal decompositions of the spaces H and F :

$$H = \overline{R(A^*)} \oplus Ker\, A,\ F = \overline{R(A)} \oplus Ker\, A^*. \tag{3}$$

Further, for any two closed subspace L and M of a Hilbert space H, the next decomposition holds:

$$(L \cap M)^{\perp} = \overline{L^{\perp} + M^{\perp}},\ H = \overline{L^{\perp} + M^{\perp}} \oplus (L \cap M). \tag{4}$$

Lemma 2.1. *For the bounded linear operators A, B from Hilbert space H to Hilbert spaces F and G, and for $c \in H$, $Cu = \langle c, u\rangle$, the following decompositions are true:*

$$H = \overline{R(A^*)} \oplus \mathcal{L}((I - P)c) \oplus \overline{R(B_1^*)} \oplus (Ker\, A \cap Ker\, B \cap Ker\, C), \tag{5}$$

$$H = \overline{R(B^*)} \oplus \mathcal{L}((I - Q)c) \oplus \overline{R(A_1^*)} \oplus (Ker\, A \cap Ker\, C), \tag{6}$$

$$Ker\, B = \overline{R(A_B^*)} \oplus (Ker\, A \cap Ker\, B), \tag{7}$$

Proof. Using the decompositions (4) and (3) we obtain

$$\begin{aligned} H &= \overline{(Ker\, A)^{\perp} \oplus (Ker\, C)^{\perp}} \oplus (Ker\, A \cap Ker\, C) \\ &= \overline{R(A^*)} \oplus \mathcal{L}(c) \oplus (Ker\, A \cap Ker\, C) \\ &= \overline{R(A^*)} \oplus \mathcal{L}((I-P)c) \oplus (Ker\, A \cap Ker\, B \cap Ker\, C). \end{aligned}$$

Similarly, applying (3) to $B_1 : Ker\, A \cap Ker\, C \to G$ we obtain

$$Ker\, A \cap Ker\, C = \overline{R(B_1^*)} \oplus (Ker\, A \cap Ker\, B \cap Ker\, C).$$

Hence, we have proved the equality (5); (6) can be proved in a similar way.

The next lemma is related to normally solvable operators.

Let us remark that an operator $A : H \mapsto F$ is said to be normally solvable if $R(A) = \overline{R(A)}$. This is equivalent to $R(A^*) = \overline{R(A^*)}$. (s. [10], pp. 153.)

Lemma 2.2. ([10], pp. 153) *A bounded linear operator $A : H \to F$ is normally solvable if and only if*

$$\mu := \inf\{\|Au\| : u \perp Ker\, A, \|u\| = 1\} > 0.$$

The immediate consequence of this Lemma is the following.

Lemma 2.3. ([10], pp. 153) *If linear operator $A : H \to F$ is not normally solvable, then there exists a sequence (p_n) such that $p_n \in \overline{R(A^*)}$, $\|p_n\| = 1$ and $Ap_n \to 0$ as $n \to \infty$.*

The restriction of a normally solvable operator $A : H \to F$ to the subspace $R(A^*)$ is invertible, so there exists $M > 0$ such that

$$(\forall x \in R(A^*))\ \|x\| \le M\|Ax\|. \tag{8}$$

If $A(V)$ is closed for a closed set $V \subseteq H$, then the inverse image

$$A^{-1}(AV) = Ker\, A + V$$

is closed set. If A is a normally solvable operator, then the converse statement is also true: if $Ker\, A + V$ is aclosed set, then the set $A(V)$ is also closed.

Now, it is easy to prove that for a normally solvable operator A and for a closed subspace $M \subseteq H$ of a finite codimension, we have $A(M) = \overline{A(M)}$.

Indeed, $codim\, M < +\infty$, implies that $dim\, M^{\perp} < +\infty$. If we denote the operator of orthogonal projection from H onto subspace M^T by T, then it is easy to prove the equality $M + Ker\, A = M + T(Ker\, A)$. From $T(Ker\, A) \subseteq M^{\perp}$, it follows that $dim(T(Ker\, A) < +\infty$. So, we get that the set $M + Ker\, A$ is closed. Normal solvability of the operator A implies that $A(M) = \overline{A(M)}$. □

Lemma 2.4. *Let L and M be closed subspaces of a Hilbert space H. If $dim\, L < +\infty$, then $A(M) = \overline{A(M)}$ if and only if $A(L \cap M) = \overline{A(L \cap M)}$.*

Proof. From $codim\, L < +\infty$ it follows that there exist $h_1, \ldots, h_n$ in H, such that $L^{\perp} = \mathcal{L}(h_1, \ldots, h_n)$, i.e.

$$H = \mathcal{L}(h_1, \ldots, h_n) \oplus L.$$

As earlier, let us denote the operator of orthogonal projection onto $M^{\perp}$ by T Note that

$$M^{\perp} \oplus \mathcal{L}(h_1, \ldots, h_n) \; = \; M^{\perp} \oplus \mathcal{L}((I - T)h_1, \ldots, (I - T)h_n).$$

Applying (4) we obtain

$$\begin{aligned} H = \; & (M^{\perp} \oplus \mathcal{L}(h_1, \ldots, h_n)) \oplus M \cap L = \\ & M^{\perp} \oplus \mathcal{L}((I - T)h_1, \ldots, (I - T)h_n) \oplus M \cap L. \end{aligned}$$

This equality and decomposition $H = M \oplus M^{\perp}$ imply that

$$M = \mathcal{L}((I - T)h_1, \ldots, (I - T)h_n) \oplus (M \cap L). \tag{9}$$

If $A(M \cap L) = \overline{A(M \cap L)}$ then, using (9), we obtain $A(M) = \overline{A(M)}$. Now, let us assume that $A(M) = \overline{A(M)}$. It means that the restriction of the operator A to the subspace M is a normally solvable operator. From (9), we can conclude that $M \cap L$ is a closed subspace of a finite codimension in the subspace M. Hence, $A(L \cap M)$ is a closed subspace of the space H.

Lemma 2.5. *If $Int\, U = \emptyset$, $U = \{u \in H : \|Bu\| \leq r, \langle c, u\rangle \leq \beta\}$, then*

(i) $U = S := \{u \in H : \|Bu\| = r, \langle c, u\rangle = \beta\}$;
(ii) $Ker\, B \subseteq Ker\, C$, *where* $Cu = \langle c, u\rangle$;
(iii) $(\forall u \in U)\, U = u + Ker\, B$.

Proof. (i) If $\|Bv\| < r$ and $v \in U$, then there exists an open set $V(v)$ containing v, such that $\|Bx\| < r$ for every $x \in V(v)$. In this case, having in mind that $Int\, U = \emptyset$, we can conclude that $\langle c, v\rangle = \beta$. Then there exists a point $x_0 \in V(v)$ such that $\langle c, x_0\rangle < \beta$. This contradicts $Int\, U = \emptyset$. So, we have $\|Bv\| = r$ for every $v \in U$. Similarly, we can prove that $\langle c, v\rangle = \beta$ for every $v \in U$. Hence, $U = S$.

(ii) We shall prove that $(I - Q)c = 0$, where Q is the orthogonal projection onto $\overline{R(B^*)}$. Let us assume the converse. Then a point $v = u + \gamma(I - Q)c, u \in U = S$, $\gamma < 0$, satisfies the conditions $\|Bv\| = r$ and $\langle c, v\rangle < \beta$. Since $U = S$, we have a contradiction. Hence, $(I - Q)c = 0$, i.e. $c \in R(B^*) \perp Ker\, B$. It immediately implies the inclusion $Ker\, B \subseteq Ker\, C$.

(iii) Let x and u be arbitrary points from $U = S$. Then $\langle c, x - u\rangle = \langle c, x\rangle - \langle c, u\rangle = 0$. Hence, $x - u \in Ker\, C$. Since, $U = S$ is a convex set, it follows that $\|B(\alpha u + (1 - \alpha x)\| = r$ for any $\alpha \in [0, 1]$. This implies that $\langle Bx, Bu\rangle = r^2$. Thus we have $\|B(x - u)\| = r^2 - 2r^2 + r^2 = 0$, i.e. $x - u \in Ker\, B$. Therefore, we proved the inclusion $U \subseteq u + Ker\, B \cap Ker\, C$. The converse inclusion is trivial. Now, (iii) follows from $U = u + Ker\, B \cap Ker\, C$ and (ii).

Lemma 2.6. *If there exists $u \in S$ such that $B^*Bu \in \mathcal{L}(c)$ and $\beta < 0$, then $Int\, U = \emptyset$.*

Proof. Suppose that $B^*Bu = \alpha c, \alpha \neq 0$. Multiplying this equality by u, we obtain $r^2 = \|Bu\|^2 = \alpha\langle c, u\rangle = \alpha\beta$. Since $\beta < 0$, it follows that $\alpha < 0$.

Assume that $Int\, U \neq \emptyset$. Then there exists $v \in U$ such that $\|Bv\| < r$ and $\langle c, v\rangle < \beta$. Now it follows that $\langle Bu, Bv\rangle \leq \|Bu\| \cdot \|Bv\| < r^2$. We obtained the contradiction that proves Lemma.

3. Convergence of minimizing sequences and existence of solutions

It is clear that the problem (1) has a solution if and only if the projection $P_r(f)$ of f on $\overline{A(U)}$ belongs to $A(U)$. Having in mind that $P_r(F) = \overline{A(U)}$, we can conclude that the problem (1) has a solution for every $f \in F$ if and only if $A(U) = \overline{A(U)}$.

Note that convexity and continuity of the function J imply its lower weakly semi-continuous. The set U is weakly closed, because it is convex and closed. Now, it is easy to prove that if for any $f \in F$ there exists at least one minimizing sequence (u_n), then, for such an f, problem (1) has a solution.

Indeed, then there exist a subsequence (u_{n_k}) of (u_n) and a point $u_* \in H$, so that (u_{n_k}) weakly converges to u_*. Since the set U is weakly closed, $u_* \in U$. Both this and lower semi-continuous of J imply

$$J(u_*) \leq \liminf J(u_{n_k}) = J_*.$$

Hence, $J(u_*) = J_*$, i.e. $u_* \in U_*$.

Note that if $U_* \neq \emptyset$, then for every $u_* \in U_*$,

$$U_* = (u_* + Ker\, A) \cap U.$$

From $J(u) = J(v) + \langle J'(v), u - v\rangle + \|A(u-v)\|^2$ and from optimality criterion of the element $u_* \in U_*$ (s. [11], p. 161, Theorem 3) $(\forall u \in U)\langle J'(u_*), u - u_*\rangle \geq 0$, we have $\|Au - Au_n\|^2 \leq J(u) - J(u_*)$.

This implies that $Au_n \to Au_*$ as $n \to \infty$, for every minimizing sequence (u_n).

If operator A is normally solvable, then, from (3) and (8) (for operator P of orthogonal projection from H to $\overline{R(A^*)}$) we have

$$\|P(u_n - u_*)\| \leq m\|AP(u_n - u_*)\| = \|Au_n - Au_*\| \to 0 \text{ as } n \to \infty$$

i.e. $Pu_n \to Pu_*$ as $n \to \infty$.

Theorem 3.1. *Suppose the following conditions hold:*
(i) A is a normally solvable operator;
(ii) $B(Ker\, A)$ is a closed subspace of G..
Then the set U_* of the solutions of the problem (1), (2) is nonempty and every minimizing sequence converges to U_*.

Proof. First, let us prove that for each $f \in F$ there exists a bounded minimizing sequence of this problem.

Condition (ii) and Lemma 2.4 imply that the operator B_1 is also normally solvable. Therefore, the equality (5) can be written as

$$H = R(A^*) \oplus \mathcal{L}((I-P)c) \oplus R(B_1^*) \oplus (Ker\, A \cap Ker\, B \cap Ker\, C)$$

where $C(u) = \langle c, u\rangle$. The elements of a minimizing sequence (u_n) can be decomposed in the following way

$$\begin{aligned} u_n = \;& Pu_n + \gamma_n(I-P)c + b_n^* + b_n, \\ & \gamma_n \in \mathbb{R}, b_n^* \in R(B_1^*), b_n \in Ker\, A \cap Ker\, B \cap Ker\, C. \end{aligned}$$

Observe the minimizing sequence

$$w_n = Pu_n + \gamma_n(I-P)c + b_n^*$$

and note that (i) implies

$$Pu_n \to Pu_* \text{ as } n \to \infty.$$

Let us consider two cases.

(I) If the sequence (γ_n) is bounded or $(I-P)c = 0$, then we can assume that $\gamma_n \to \gamma_* \in R$ as $n \to \infty$. It is obvious that in this case the sequences (Pu_n) and (b_n^*) are bounded. Hence, the minimizing sequence (w_n) is also bounded. Consequently, the set U_* of the solutions of the given problem is nonempty. Since $b_n^* + b_n \perp c$ we have

$$\langle Pu_* + \gamma_*(I-P)c, c\rangle = \lim_{n\to\infty} \langle Pu_n + \gamma_n(I-P)c, c\rangle = \lim_{n\to\infty} \langle w_n, c\rangle \leq \beta$$

The sequence $v_n = Pu_* + \gamma_*(I-P)c + b_n^* + b_n$ satisfies the inequality $\langle v_n, c\rangle \leq \beta$.

Then, we have to consider two possibilities: (a) $\|Bv_n\| \leq r$, and (b) $\|Bv_n\| > r$ (otherwise, we can consider a subsequence of (v_n).)

If $\|Bv_n\| \leq r$, then $v_n \in U_*$. Therefore

$$d(u_n, U_*) \leq \|u_n - v_n\| \leq \|Pu_n - Pu_* + (\gamma_n - \gamma_*)(I-P)c\| \to 0,\ n \to \infty.$$

In case of $\|Bv_n\| > 0$, we have that

$$\begin{aligned} r < \|Bv_n\| &\leq \|B(v_n - w_n)\| + \|Bw_n\| \\ &\leq \|B(Pu_n - Pu_* + (\gamma_n - \gamma_*)(I-P)c)\| + r, \end{aligned}$$

what implies that

$$\lim_{n\to\infty} \|Bv_n\| = r \text{ and } \lim_{n\to\infty} \|Bw_n\| \leq r. \tag{10}$$

Since the sequence (b_n^*), $b_n^* \in R(B_1^*)$ is bounded, we can assume that (b_n^*) converges weakly to $b_0^* \in R(B_1^*)$ as $n \to \infty$. Then, the minimizing sequence (w_n) converges weakly to $w_* = Pu_* + \gamma_*(I-P)c + b_0^* \in U_*$.

Within the scope of this case, we will again consider two possibilities:

If $\|Bw_*\| = r$, then

$$\|B(b_n^* - b_0^*)\|^2 = \|B(w_n - w_* + Pu_* - Pu_n + (\gamma_n - \gamma_*)(I-P)c\|^2, \tag{11}$$

implies that $\|B(b_n^* - b_0^*)\| \to 0$ as $n \to \infty$. From here on, having in mind that B_1 is normal solvable, we can conclude that (b_n^*) converges (strongly) to b_0^* as $n \to \infty$. Then $w_n \to w_*$ as $n \to \infty$, and

$$d(u_n, U_*) \leq \|u_n - (w_* + b_n)\| = \|w_n - w_*\| \text{ as } n \to \infty.$$

If $\|Bw_*\| < r$, then

$$\lim_{n\to\infty} \|B(b_n^* - b_0^*)\|^2 = r^2 - \|Bw_*\|^2 > 0.$$

For each $n \in N$, there exists $\alpha_n > 0$ such that $\|B(w_* + \alpha_n(b_n^* - b_0^*))\|^2 = r^2$. Now, using the last two relations, it is easy to prove $\lim_{n\to\infty} \alpha_n = 1$. Sequence

$$x_n = w_* + \alpha_n(b_n^* - b_0^*) + b_n + b_0^* = Pu_* + \gamma_*(I-P)c + \alpha_n b_n^* + (1-\alpha_n)b_0^* + b_n$$

satisfies the following conditions

$$Ax_n = Aw_*,\ \|Bx_n\| = r, \langle c, x_n\rangle = \langle c, w_*\rangle \le \beta,$$

and so $x_n \in U_*$ for every $n \in N$. Then

$$\begin{aligned} & d(u_n, U_*) \le \|u_n - x_n\| \\ & = \|Pu_n - Pu_* + (\gamma_n - \gamma_*)(I-P)c + (1-\alpha_n)(b_n^* - b_0^*)\| \to 0 \end{aligned}$$

as $n \to \infty$. Therefore, in case of (I), every minimizing sequence of the problem (1), (2) converges to the set of its solutions.

(II) Now, assume the sequence (γ_n) is unbounded and $(I-P)c \neq 0$. Then for $B(I-P)c \in G$ there exist $p_0 \in R(B_1^*)$ and $q_0 \in Ker\, B_1^*$, such that

$$B(I-P)c = Bp_0 + q_0.$$

Since $B(\gamma_n p_0 + b_n^*) \perp q_0$, we have

$$\begin{aligned} \|B(\gamma_n(I-P)c + b_n^*)\|^2 &= \|B(\gamma_n p_0 + b_n^*) + \gamma_n q_0\|^2 \\ &= \|B(\gamma_n p_0 + b_n^*)\|^2 + \gamma_n^2 \|q_0\|^2. \end{aligned}$$

The last equality implies that $q_0 = 0$. Then $B((I-P)c - p_0) = 0$, and consequently, there is $z_0 \in Ker\, B$, such that

$$(I-P)c = p_0 + z_0.$$

It is easy to prove that $z_0 \in Ker\, A$ and $\langle c, z_0\rangle = \|(I-P)c\|^2 \neq 0$.

Now, using the equality

$$B(\gamma_n(I-P)c + b_n^*) = B(\gamma_n p_0 + b_n^*),$$

we conclude that $\gamma_n p_0 + b_n^* \in R(B_1^*)$. Consequently, $(\gamma_n p_0 + b_n^*)$ is a bounded sequence. Observe the sequence

$$v_n = Pu_n + \gamma_n p_0 + b_n^* + \gamma_n^* z_0, \text{ where } \gamma_n^* = \frac{\beta - \langle Pu_n + \gamma_n p_0 + b_n^*, c\rangle}{\langle c, z_0\rangle}.$$

It is obvious that

$$Av_n = Aw_n, Bv_n = Bw_n, \langle c, v_n \rangle = \beta,$$

which makes (v_n) a bounded minimizing sequence of the given problem. Therefore, $U_* \neq \emptyset$.

Let us observe an arbitrary minimizing sequence (u_n). It can be written in the form

$$u_n = Pu_n + b_n^* + \gamma_n p_0 + b_n + \gamma_n z_0.$$

Let us observe the sequences

$$w_n = Pu_n + b_n^* + \gamma_n p_0 + \gamma_n z_0 \text{ and } y_n = Pu_* + \overline{b_n^*} + b_n + \delta_n z_0,$$

where

$$\delta_n = \frac{\langle u_n, c \rangle - \langle Pu_* + \overline{b_n^*}, c \rangle}{\langle z_0, c \rangle} = \frac{\langle Pu_n - Pu_*, c \rangle}{\langle z_0, c \rangle} + \gamma_n,$$

Note that $\lim_{n\to\infty}(\gamma_n - \delta_n) = 0$. The numbers δ_n have been chosen in such way that $\langle y_n, c \rangle \leq \beta$. If $\|By_n\| \leq r$, then $y_n \in U_*$ and therefore

$$d(u_n, U_*) \leq \|u_n - y_n\| = \|Pu_n - Pu_* + (\gamma_n - \delta_n)z_0\| \to 1 \text{ as } n \to \infty.$$

If $\|By_n\| > r$, then following the previous procedure, we obtain

$$\lim_{n\to\infty} \|By_n\| = r, \ \lim_{n\to\infty} \|Bw_n\| = r,$$

and

$$\overline{b_n^*} = b_n^* + \gamma_n p_0 \rightharpoonup \overline{b_0^*} \in R(B_1^*)\text{as } n \to \infty.$$

It follows that

$$Pu_n + \overline{b_n^*} \rightharpoonup \overline{w_*} = \overline{b_0^*} + Pu_* \text{ as } n \to \infty, \text{ and } \|B\overline{w_*}\| \leq r.$$

Again we need to consider two possibilities: $\|B\overline{w_*}\| = r$ and $\|B\overline{w_*}\| < r$.

If $\|B\overline{w_*}\| = r$, then, just like in the previous part of the proof, we can prove the strong convergence of the sequence $\overline{b_n^*}$ to $\overline{b_0^*}$ as $n \to \infty$. Observe the sequence $z_n = \overline{w_*} + \overline{b_n} + \delta_n^* z_0$ where

$$\delta_n^* = \frac{\langle u_n, c \rangle - \langle \overline{w_*}, c \rangle}{\langle z_0, c \rangle}.$$

Then $\langle z_n, c \rangle \leq \beta$ and

$$\delta_n^* - \gamma_n = \frac{\langle Pu_n - Pu_* + \overline{b_n^*} - \overline{b_0^*}, c \rangle}{\langle z_0, c \rangle} \to 0 \text{ as } n \to \infty.$$

Besides, $Az_n = A\overline{w_*},\ \|Bz_n\| = \|B\overline{w_*}\| = r$, such that $z_n \in U_*$. Now we have

$$d(u_n, U_*) \le \|u_n - z_n\| = \|Pu_n - Pu_* + \overline{b_n^*} - \overline{b_0^*} + (\gamma_n - d_n^*)z_0\| \to 0 \text{ as } n \to \infty.$$

Finally, we have to consider the case of $\|B\overline{w_*}\| < r$. Then

$$\lim_{n\to\infty} \|B(\overline{b_n^*} - \overline{b_0^*})\|^2 = r^2 - \|B\overline{w_*}\|^2 > 0.$$

If $\alpha_n, n = 1, 2, \dots$ are numbers chosen such that

$$\|B(\overline{w_*} + \alpha_n(\overline{b_n^*} - \overline{b_0^*}))\|^2 = r^2 \text{ with } \lim_{n\to\infty} \alpha_n = 1,$$

and (s_n) is the sequence defined by

$$s_n = \overline{w_*} + \alpha_n(\overline{b_n^*} - \overline{b_0^*}) + b_n + r_n z_0 = Pu_* + \alpha_n\overline{b_n^*} + (1-\alpha_n)\overline{b_0^*} + b_n + \eta_n z_0,$$

where

$$\eta_n = \frac{\langle u_n, c\rangle - \langle Pu_* + \alpha_n\overline{b_n^*} + (1-\alpha_n)\overline{b_0^*}, c\rangle}{\langle z_0, c\rangle},$$

then $\langle s_n, c\rangle \le \beta$ and

$$\eta_n - \gamma_n = \frac{\langle Pu_n - Pu_* + (1-\alpha_n)(\overline{b_n^*} - \overline{b_0^*}), c\rangle}{\langle z_0, c\rangle} \to 0 \text{ as } n \to \infty.$$

Besides, $As_n = A\overline{w_*},\ \|Bs_n\| = \|B\overline{w_*}\| = r$, so that $s_n \in U_*$. Therefore

$$d(u_n, U_*) \le \|u_n - s_n\| = \|Pu_n - Pu_* + (1-\alpha_n)(\overline{b_n^*} - \overline{b_0^*}) + (\gamma_n - \eta_n)z_0\| \to 0$$

as $n \to \infty$. This completes the proof.

Example. Take $H = F = G = l_2$ and consider two closed subspaces of l_2 :

$$L = \{x \in l_2 :\ x = (0, x_2, 0, x_4, 0, x_6, 0, \dots)\},$$

$$M = \{x \in l_2 :\ x = (0, x_2, \frac{x_2}{2}, x_4, \frac{x_4}{4}, x_6, \frac{x_6}{6}, \dots)\}.$$

Define A as the orthoprojector of l_2 onto $L^\perp$ and B as the orthoprojector of l_2 onto $M^\perp$, $c = 0 \in H, \beta = 0$. Then $A = A^*,\ B = B^* = B^*B,\ \operatorname{Ker} A = L,\ \operatorname{Ker} B = M$, operators A and B are normally solvable but both relations (i) and (ii) from Theorem 3.1 are violated:

$$x_0 = (1, 0, 0, \dots) \in R(A^*) \cap R(B^*B) = L^\perp \cap M^\perp \ne \{0\},$$

$$\mathrm{Ker}\,A + \mathrm{Ker}\,B = L + M \neq \overline{L+M} = \overline{\mathrm{Ker}\,A + \mathrm{Ker}\,B} = \{x_0\}^{\perp}.$$

It means that in this case the problem (1),(2) can not have a solution for every $f \in l_2$.

The next theorem shows that if the first condition of the previous theorem is violated, then the problem (1), (2) will not be well-posed anymore.

Theorem 3.2. *Suppose that*
(i) $\overline{R(A)} \neq R(A)$;
(ii) $U_* \cap Int\{u \in H : \|Bu\| \leq r\} \neq \emptyset$.

Then there exists a minimizing sequence of the problem (1), (2) that does not converge to the set U_* *of its solutions.*

Proof. The condition (i), according to Lemma 2.3, implies the existence of a sequence (p_n) such that

$$p_n \in \overline{R(A^*)},\ \|p_n\| = 1,\ \lim_{n\to\infty} Ap_n = 0.$$

Since $U_* \cap Int\,U_1 \neq \emptyset$, we can infer that there is an element $u_* \in U_*$ such that $\|Bu_*\| < r$. Choose an $\varepsilon_0 > 0$ such that $\|B(u_* \pm \varepsilon_0 p_n)\| < r$. Consider the sequence (v_n):

$$v_n = \begin{cases} u_* + \varepsilon_0 p_n, & \text{if } \langle p_n, c\rangle \leq 0 \\ u_* - \varepsilon_0 p_n, & \text{if } \langle p_n, c\rangle > 0 \end{cases}.$$

Hence, $v_n \in U$ and sequence (v_n) is minimizing. Since $U_* = \{u_* + Ker\,A) \cap U$, it follows that for every $v_* \in U_*$ there exists $x(v_*) \in Ker\,A$ such that $v_* = u_* + x(v_*)$. From

$$\|v_n - v_*\|^2 = \|u_* \pm \varepsilon_0 p_n - u_* - x(v_*)\|^6 = \varepsilon_0^2 + \|x(v_*)\|^2 \geq \varepsilon_0^2$$

it follows that the sequence $(d(u_n, U_*))$ does not converge to 0 as $n \to \infty$. This completes the proof of the Theorem.

Theorem 3.3. *Suppose that the next conditions are satisfied:*
(i) $B : H \to G$ *is normal solvable bounded linear operator;*
(ii) $A(Ker\,B) = \overline{A(Ker\,B)}$.
Then the set U_* *of the solutions of the problem (1), (2) is nonempty.*
If, in addition, $U_* \subseteq \{u \in H : \|Bu\|^2 = r^2, \langle c, u\rangle = \beta\}$, *then every minimizing sequence of the problem (1), (2), converges to the set of its solutions.*

Proof. We will prove only the convergence of minimizing sequences (second part of Theorem).

Lemma 2.4 and condition (ii) imply that the operator A_1 is normally solvable. Taking into account the condition (i) the equality (6) can be written as

$$H = \overline{R(B^*)} \oplus \mathcal{L}((I-Q)c) \oplus \overline{R(A_1^*)} \oplus (Ker\, A \cap Ker\, B \cap Ker\, C),$$

The elements of the minimizing sequence (u_n) can be written in the form

$$u_n = x_n + \gamma_n (I-Q)c + a_n^* + a_n,$$

where

$$x_n \in R(B^*),\ \gamma_n \in R, a_n^* \in R(A_1^*),\ a_n \in Ker\, A \cap Ker\, B \cap Ker\, C.$$

From condition (i), relations $\|Bu_n\| = \|Bx_n\| \le r$, by applying (8) to the operator B, we can conclude that the sequence (x_n) is bounded. We can assume that this sequence weakly converges to some $x_0 \in R(B^*)$ as $n \to \infty$.

(I) Suppose that the sequence (γ_n) is bounded or that $(I-Q)c = 0$. In both cases we can assume that $\gamma_n \to \gamma_* \in R$ as $n \to \infty$. Since (u_n) is a minimizing sequence, it follows that

$$(\forall u_* \in U_*) A u_* = \lim_{n\to\infty} A u_n = \lim_{n\to\infty} A(x_n + \gamma_n (I-Q)c + a_n^*). \qquad (12)$$

Since the sequence $(x_n + \gamma_n (I-Q)c)$ is bounded and the operator A_1 is normally solvable, by applying the relation (8) to A_1, we obtain that the sequence (a_n^*) is bounded. We can assume that (a_n^*) weakly converges to a_0^* as $n \to \infty$. The sequence $(v_n), v_n = x_n + \gamma_n (I-Q)c + a_n^*$ is also minimizing and (v_n) weakly converges to $v_* = x_0 + \gamma_* (I-Q)c + a_0^*$ as $n \to \infty$. It is clear that $v_* \in U_*$. Then, using (iii), we have $r = \|Bv_*\| = \|Bx_0\|$. Having in mind the fact that $\|Bx_0\| = r$ and that the sequence (x_n) weakly converges to $x_0 \in R(B^*)$ as $n \to \infty$, similar to the proof of (a_{21}) of Theorem 3.1, we can prove that (x_n) converges to x_0. Further, using (12), we have that $\lim_{n\to\infty} A a_n^* = A a_0^*$. Since $a_n^* - a_0^* \in R(A_1^*)$, applying the relation (8) to the operator A_1, we obtain that the sequence (a_n^*) converges to a_0^* as $n \to \infty$.

Now, let us observe the sequence $(w_n), w_n = x_0 + \gamma_* (I-Q)c + a_0^*$. We can note that $w_n \in U_*$ for every $n \in N$. Then

$$d(u_n, U_*) \le \|u_n - w_n\| = \|x_n - x_0 + (\gamma_n - \gamma_*)(I-Q)c + a_n^* - a_0^*\| \to 0$$

as $n \to \infty$. Hence, in the case (a) the problem is wellposed.

(II) Now, suppose the sequence (γ_n) be unbounded and $(I-P)c \neq 0$. In the similar way as in the proof of (11), it is possible to prove that

$$(I-P)c = p_0 + z_0,\ \langle z_0, c\rangle \neq 0,\ p_0 \in R(A_1^*),\ z_0 \in Ker\, A \cap Ker\, B.$$

We can write the elements of the minimizing sequence (u_n) in the following form

$$u_n = x_n + \overline{a_n^*} + a_n + \gamma_n z_0,$$

where $\overline{a_n^*} = \gamma_n p_0 + a_n^* \in R(A_1^*)$. We have already proved that the sequence (x_n) weakly converges to $x_0 \in R(B^*)$ as $n \to \infty$. Then from

$$\lim_{n\to\infty} A(x_n + \overline{a_n^*}) = Au_*,\ u_* \in U_*,$$

Taking into account that the operator A_1 is normally solvable, and applying (8) to A_1, we obtain that the sequence $(\overline{a_n^*})$ is bounded. We can assume that the whole sequence $(\overline{a_n^*})$ weakly converges to $\overline{a_0^*} \in R(A_1^*)$. Hence, the sequence $(w_n), w_n = x_n + \overline{a_n^*}$, weakly converges to $w_* = x_0 + \overline{a_0^*}$. Since, $Aw_n = Au_n \to Au_*$ as $n \to \infty$, it follows that $Aw_* = Au_*$. In addition, $\|Bw_n\| = \|Bu_n\| \leq r$ implies $\|Bw_*\| \leq r$. Defining a real sequence (r_n) by

$$r_n = \frac{\langle u_n, c\rangle - \langle w_*, c\rangle}{\langle z_0, c\rangle} + \gamma_n = \frac{\langle x_n - x_0 + \overline{a_n^*} - \overline{a_0^*}, c\rangle}{\langle z_0, c\rangle} + \gamma_n,$$

it follows that

$$r_n - \gamma_n = \frac{\langle u_n, c\rangle - \langle w_*, c\rangle}{\langle z_0, c\rangle} = \frac{\langle x_n - x_0 + \overline{a_n^*} - \overline{a_0^*}, c\rangle}{\langle z_0, c\rangle} + \gamma_n \to 0 \text{ as } n \to \infty.$$

Then the sequence $(z_n), z_n = w_* + r_n z_0 + a_n$ satisfies the conditions:

$$\langle z_n, c\rangle = \langle c, u_n\rangle \leq \beta,\ Bz_n = Bw_*,\ Az_n = Aw_* = Au_*.$$

Hence, $z_n \in U_*$ for every $n \in N$. Now, using the condition (iii), as in the proof of the first part of the Theorem, we can prove that

$$\lim_{n\to\infty} x_n = x_0,\ \lim_{n\to\infty} \overline{a_n^*} = \overline{a_0^*}.$$

Finally, we have

$$d(u_n, U_*) \leq \|u_n - z_n\| = \|x_n - x_0 - \overline{a_n^*} - \overline{a_0^*} + (\gamma_n - r_n)z_0 - a_n^* - a_0^*\| \to 0$$

as $n \to \infty$. This completes the proof of Theorem. $\square$

If condition (i) in Theorem 3.3 is not satisfied, then the problem (1),(2) will not be wellposed.

Theorem 3.4. *Suppose that the following conditions are satisfied:*
(i) $\overline{A(Ker\,B)} \neq A(Ker\,B)$*;*
(ii) $U_* \subseteq S$*.*
Then there exists a minimizing sequence of the problem (1), (2) that does not converge to the set U_* *of its solutions.*

Proof. First, let us note that the condition (ii) implies the equality (34). Further, Lemma 2.4 and the condition (i) imply that $R(A_1^*) \neq \overline{R(A_1^*)}$. Applying Lemma 2.3 to the operator A_1, we conclude that there exists a sequence (p_n), such that

$$p_n \in \overline{R(A_1^*)},\ \|p_n\| = 1, \lim_{n\to\infty} Ap_n = 0.$$

Observe the sequence $u_n = u_* + p_n$. Since $p_n \in \overline{R(A_1^*)} \subseteq Ker\,A \cap Ker\,B$, we obtain $Bu_n = Bu_*$ and $\langle c, u_n\rangle = \langle c, u_*\rangle = \beta$. The elements u_n satisfies the condition $\lim_{n\to\infty} Au_n = Au_* + \lim_{n\to\infty} Ap_n = Au_*$. Hence, (u_n) is a minimizing sequence. Then for every

$$v_* = u_* + x(v_*) \in U_* = u_* + Ker\,A \cap Ker\,B \cap Ker\,C,$$

we have

$$\|u_n - v_*\|^2 = \|p_n - x(v_*)\|^2 = \|p_n\|^2 + \|x(v_*)\|^2 \geq 1.$$

Hence, the sequence $d(u_n, U_*))$ does not converge to 0. This completes the proof of the theorem.

The next theorem gives necessary and sufficient conditions for convergence of minimizing sequences of one class of minimization of quadratic function.

Theorem 3.5. *If* $Int\,U = \emptyset$ *then every minimizing sequence of the problem (1), (2) converges to the set* U_* *of its solutions if and only if* $A(Ker\,B) = \overline{A(Ker\,B)}$.

Proof. Suppose that $A(Ker\,B)$ is a closed subspace of space F. Then the operator A_B is normally solvable and the equality (7) can be written as

$$Ker\,B = R(A_B^*) \oplus (Ker\,A \cap Ker\,B)$$

This equality and Lemma 2.5 imply that

$$(\forall u_* \in U_*)\ U = u_* + R(A_B^*) \oplus (Ker\,A \cap Ker\,B).$$

The elements of minimizing sequence (u_n) can be written in the form

$$u_n = u_* + a_n^* + b_n,$$

where

$$a_n^* \in R(A_B^*),\ b_n \in Ker\, A \cap Ker\, B.$$

Then $Au_n \to Au_*$ as $n \to \infty$, implies that $Aa_n^* \to 0$ as $n \to \infty$. Since a_n^* belongs to $R(A_B^*)$, applying (8) to the operator A_B, we obtain that $a_n^* \to 0$ as $n \to \infty$.

Further, by applying Lemma 2.5, it is easy to prove that

$$(\forall u_* \in U_*)\ U_* = u_* + (Ker\, A \cap Ker\, B).$$

Observe the sequence (v_n) defined by $v_n = u_* + b_n \in U_*$. Then

$$d(u_n, U_*) = \|u_n - v_n\| = \|a_n^*\| \to 0 \text{ as } n \to \infty.$$

Now, assuming that every minimizing sequence of the problem (1), (2) converges to the set U_* we shall prove that $A(Ker\, B) = \overline{A(Ker\, B)}$. Otherwise, we would have $R(A_B^*) \neq \overline{R(A_B^*)}$. Lemma 2.3 implies that there exists a sequence (p_n) such that

$$p_n \in \overline{R(A_B^*)},\ \|p_n\| = 1,\ \lim_{n\to\infty} Ap_n = 0.$$

Then $(v_n), v_n = u_* + p_n$ is a minimizing sequence. Moreover, every element $v_* \in U_* = u_* + (Ker\, A \cap Ker\, B)$ can be written as $v_* = u_* + x(v_*)$, where $x(v_*) \in Ker\, A \cap Ker\, B$. Now, we have

$$\|v_n - v_*\| = \|p_n - x(v_*)\| = \|p_n\|^8 + \|x(v_*)\|^2 \geq 1.$$

It means that the sequence $(d(u_n, U_*))$ does not converge to 0. However this is in opposition to our assumption that every minimizing sequence of the problem (1), (2) converges to U_*. Thus, $A(Ker\, B) = \overline{A(Ker\, B)}$. $\square$

Let us note that for $C = 0, \beta \geq 0$, we will obtain the corresponding theorems related to problem of minimization of quadratic function on the set $\{u \in H : \|Bu\|^2 \leq r^2\}$, and as for $B = 0$, we will obtain the theorems related to the problem of minimization of quadratic function on a half-space $\{u \in U : \langle c, u\rangle \leq \beta\}$.

4. Stabilization of minimizing sequences

Let us consider problem of minimization of quadratic function $J(u) = \|Au - f\|^2$ on the set $U = \{u \in H : \langle c, u\rangle \leq \beta\}$.

In practice, instead of the exact operator A and the elements $f,\ c$, we deal with their approximations $A_\mu \in \mathcal{L}(H, F),\ f_\delta \in F$, and $c_\sigma \in H$, such that

$$\|A - A_\mu\| \leq \mu, \ \ \|f - f_\delta\| \leq \delta, \ \ \|c - c_\sigma\| \leq \sigma,$$

where $\mu,\ \delta$ and σ are small positive real numbers.

Generally speaking, this problem is unstable with respect to the perturbations of the initial data A, f, c and it requires application of regularization method [6],[8], [9], [10], [12].

In further text, we will suppose that the sets of solutions of the given problems and of the corresponding approximate problems are not empty. Let us denote the solution of the source problem with minimal norm by u_∞.

According to the optimality conditions, we have that u_∞ satisfies the operator equality,

$$A^*Au_\infty = A^*f, \tag{13}$$

as for the element u^* there exists $\lambda^* \geq 0$ such that

$$A^*Au_* - A^*f + \lambda^*c = 0 \tag{14}$$

$$\lambda^*(\langle c, u_*\rangle - \beta) = 0 \tag{15}$$

In according with Tikhonov's idea of regularization of unstable problem, we can take the (unique) solution of the problem

$$Q_\alpha(u) = \|A_\mu u - f_\delta\| + \alpha\|u\|^2 \to \inf, \ \ u \in H \tag{16}$$

for small positive real number $\alpha = \alpha(\delta, \mu),\ \delta,\ \mu$, as an approximation of a solution of our problem . Let us note that the solution of the problem (16) can be represented by

$$u_\alpha = \overline{g_\alpha}(A_\mu^* A_\mu) A_\mu^* f_\delta,$$

where $\overline{g_\alpha}(t) = (\alpha + t)^{-1}$. We can say that the method (16) is generated by the system of functions $\{\overline{g_\alpha}\}$.

The generalizations of the previous method regarding the problem (1), (2) with $B = 0$, were observed in [10]. These generalized methods were generated with the system of continuous functions $g_\alpha : [0, a] \to R, \ \ \alpha > 0$, such that

$$(\forall t \in [0, a])(\forall \alpha > 0)\, 1 - tg_\alpha(t) \geq 0 \tag{17}$$

$$\sup\{t^p(1-tg_\alpha(t)) : t\in[0,a]\} \le \gamma_p\alpha^p,\ (0\le p\le p_0,\ p_0>0, \gamma_p\equiv \text{const}) \quad (18)$$

Real number p_0 is called the qualification of the system $\{g_\alpha\}$. The functions $\overline{g_\alpha}(t) = (\alpha+t)^{-1}$ satisfy the conditions (17)-(18) with $p_0 = 1$. However, the method (16) generated with the functions $\overline{g_\alpha}$ is not suitable for describing some algorithms for choosing the parameter α and for studying some iterative methods of regularization. In order to study these problems we should consider the system of functions $\{g_\alpha\}$, which satisfy the conditions (17)-(18) with the qualification $p_0 > 1$.

Notice that the regularization methods based on the functions $\{g_\alpha\}$ for the minimization problem without constraints were properly observed in [8] and [9]. We show that the similar class of the functions can be also used for the regularization of the problem (1).

4.1. Algorithms of stabilization.

It turns out that for studying the extremal problems with constraints, beside (17), (18), we need an additional condition

$$(\exists\beta>0)(\forall t\in[0,a])(\forall\alpha>0)\ \ (t+\beta\alpha)^{-1}\le g_\alpha(t)\le(\beta\alpha)^{-1} \quad (19)$$

The examples of the functions that satisfy (9) can be found in [10].

Since

$$\beta\alpha\|u-v\|^2 \le \langle g_\alpha^{-1}(A_\mu^*A_\mu)(u-v), u-v\rangle, u,v\in H,$$

it follows that the extremal problem

$$T_\alpha(u) =\| g_\alpha^{-\frac{1}{2}}(A_\mu^*A_\mu)u - g_\alpha^{\frac{1}{2}}(A_\mu^*A_\mu)A_\mu^*f_\delta \|^2\to \inf, u\in H$$

has the unique solution w_α. Then $T'_\alpha(w_\alpha) = 0$ i.e. $w_\alpha = g_\alpha(A_\mu^*A_\mu)A_\mu f_\delta$. We shall prove that the following estimate is true

$$\| A_\mu(w_\alpha-u_\infty) \|^2\le k(\alpha+\mu)\|w_\alpha-u_\infty\|,\ k>0. \quad (20)$$

Using the conditions $J'(u_\infty) = 0$ and $T'_\alpha(w_\alpha) = 0$, we have that

$$g_\alpha^{-1}(A_\mu^*A_\mu)w_\alpha - A_\mu^*f_\delta - A^*Au_\infty + A^*f = 0.$$

Multiplying the previous equality by $w_\alpha - u_\infty$ and using the properties of the function g_α, we obtain the inequality

$$\| A_\mu(w_\alpha-u_\infty) \|^2\le \langle (A_\mu^*A_\mu - g_\alpha^{-1}(A_\mu^*A_\mu))u_\infty, w_\alpha-u_\infty\rangle +$$

$$\langle (A^*A - A^*_\mu A_\mu)u_\infty, w_\alpha - u_\infty\rangle + \langle (A^*_\mu f_\delta - A^* f, w_\alpha - u_\infty\rangle$$

that implies the estimation (20).

In [10], Theorem 2.4, p. 100, it is proved that if parameter $\alpha = \alpha(\delta, \mu)$ satisfies

$$\alpha(\delta, \mu) \to 0,\ \frac{\mu + \delta^2}{\alpha(\delta, \mu)} \to 0,\ \ (\delta, \mu \to 0)$$

then

$$w_{\alpha(\delta,\mu)} \to u_\infty, (\delta, \mu \to 0)$$

Lemma 4.1. *Suppose that the parameter* $\alpha = \alpha(\mu)$ *is such that*

$$\alpha(\mu) \to 0, \frac{\mu}{\alpha(\mu)} \to 0,\ \ (\mu \to 0)$$

Then, for all $x \in H$, *we have*

(i) $(I - A^*_\mu A_\mu g_\alpha(A^*_\mu A_\mu))Px \to 0,$ *as* $\mu \to 0,$

(ii) $\beta\alpha g_\alpha(A^*_\mu A_\mu)x \to (I - P)x,$ *as* $\mu \to 0.$

Proof. (i) The proof of this part of Lemma can be found in [10], Lemma 2.2., p. 99.

(ii) The family of operators $\{\beta\alpha g_\alpha(A^*_\mu A_\mu)\}$ is uniformly bounded, because

$$\| \beta\alpha g_\alpha(A^*_\mu A_\mu) \| \leq \sup\{\beta\alpha g_\alpha(t) : t \in [0, a]\} \leq 1.$$

The elements $u = A^*Aw, w \in H$, generate a dense subspace in $\overline{R(A^*A)}$ and

$$\|\beta\alpha g_\alpha(A^*_\mu A_\mu)A^*Aw\| \leq \| \beta\alpha g_\alpha(A^*_\mu A_\mu)(A^*A - A^*_\mu A_\mu)w \| +$$

$$\|\beta\alpha g_\alpha(A^*_\mu A_\mu)A^*_\mu A_\mu w\| \leq k(\mu + \alpha) \to 0$$

when $\mu, \delta, \sigma \to 0$.

By virtue of Banach-Steinhaus theorem, we have

$$\beta\alpha g_\alpha(A^*_\mu A_\mu)Px \to 0, (\mu, \delta, \sigma \to 0).$$

Since, $0 \leq 1 - \beta\alpha g_\alpha(t) \leq \frac{t}{\beta\alpha}$, it follows that

$$\| (I - \beta\alpha g_\alpha(A^*_\mu A_\mu))(I - P)x \| \leq \frac{\| A^*_\mu A_\mu (I - P)x \|}{\beta\alpha} =$$

$$\frac{\| (A^*_\mu A_\mu - A^*A)(I - P)x \|}{\beta\alpha} \leq k\frac{\mu}{\alpha} \to 0\ (\mu \to 0).$$

Finally, we obtain

$$\beta\alpha g_\alpha(A_\mu^* A_\mu)x = \beta\alpha g_\alpha(A_\mu^* A_\mu)Px + \beta\alpha g_\alpha(A_\mu^* A_\mu)(I-P)x \to (I-P)x,$$

as $\mu \to 0$.

Lemma 4.2. *If the parameter $\alpha = \alpha(\mu,\sigma)$ is chosen such that*

$$\alpha(\mu,\sigma) \to 0, \frac{\mu+\sigma}{\alpha(\mu,\sigma)} \to 0 \ (\mu,\sigma \to 0)$$

then

(i) $\beta\alpha g_\alpha(A_\mu^* A_\mu)c_\sigma \to (I-P)c \ (\mu,\sigma \to 0)$

(ii) If $c = A^*Ah, h \in \overline{R(A^*A)}$, *then* $g_\alpha(A_\mu^* A_\mu)c_\sigma \to h \ (\mu,\sigma \to 0)$.

Proof. We have that

$$g_\alpha(A_\mu^* A_\mu)c_\sigma = g_\alpha(A_\mu^* A_\mu)(c_\sigma - c) + g_\alpha(A_\mu^* A_\mu)c.$$

This equality, the estimate

$$\| g_\alpha(A_\mu^* A_\mu)(c_\sigma - c) \| \leq k\frac{\mu}{\alpha},$$

and Lemma 1 imply (i).

We may also write

$$g_\alpha(A_\mu^* A_\mu)c_\sigma - h = g_\alpha(A_\mu^* A_\mu)(c_\sigma - c) + g_\alpha(A_\mu^* A_\mu)c - h =$$

$$g_\alpha(A_\mu^* A_\mu)(c_\sigma - c) + g_\alpha(A_\mu^* A_\mu)(A^*A - A_\mu^* A_\mu)h - (I - A_\mu^* A_\mu g_\alpha(A_\mu^* A_\mu)h.$$

Therefore, we obtain the inequality

$$\|g_\alpha(A_\mu^* A_\mu)c_\sigma - h\| \leq$$

$$\|g_\alpha(A_\mu^* A_\mu)\| \cdot \|c_\sigma - c\| + \|g_\alpha(A_\mu^* A_\mu)\| \cdot \|A^*A - A_\mu^* A_\mu\| \cdot \|h\| +$$

$$\|(I - A_\mu^* A_\mu g_\alpha(A_\mu^* A_\mu))h\| \leq k\left(\frac{\mu+\sigma}{\alpha} + \|I - A_\mu^* A_\mu g_\alpha(A_\mu^* A_\mu)h\|\right).$$

This inequality, Lemma 4.1 (i) and the properties of the functions g_α, imply (ii).

Taking into account the optimality conditions (13)-(15) and Lemma 4.2, it is easy to prove the following relationship between normal solutions u_* and u_∞.

Lemma 4.3. *(i) If $c \in R(A^*A)$, i.e. $c = A^*Ah$ for some $h \in \overline{R(A^*A)}$, then $u_* = u_\infty - \lambda_* h$, where*

$$\lambda^* = \begin{cases} 0, & \langle u_\infty, c\rangle \le \beta \\ \frac{\langle u_\infty, c\rangle - \beta}{\|(I-P)c\|^2}, & \langle u_\infty, c\rangle > \beta \end{cases}$$

(ii) If $c \notin R(A^*A)$, then $u_* = u_\infty - \gamma_*(I-P)c$, where

$$\gamma_* = \begin{cases} 0, & \langle u_\infty, c\rangle \le \beta \text{ or } (I-P)c = 0 \\ \frac{\langle u,_\infty, c\rangle - \beta}{\|(I-P)c\|^2}, & \langle u_\infty, c\rangle > \beta \text{ and } (I-P)c \ne 0 \end{cases}$$

As an approximation of the solution of the problem (1), one can take the element

$$u_\alpha = g_\alpha(A_\mu^* A_\mu)(A_\mu^* A_\mu f_\delta - \lambda_\alpha c_\sigma)$$

where

$$\lambda_\alpha = \begin{cases} 0, & \langle w_\alpha, c_\sigma\rangle \le \beta \\ \frac{\langle w_\alpha, c_\sigma\rangle - \beta}{\langle g_\alpha(A_\mu^* A_\mu)c_\sigma, c_\sigma\rangle}, & \langle w_\alpha, c_\sigma\rangle > \beta \end{cases}$$

and $w_\alpha = g_\alpha(A_\mu^* A_\mu)A_\mu^* f_\delta$, is the solution of the extremal problem $T_\alpha(u) \to \inf, u \in H$. The element u_α satisfies the equalities

$$T'_\alpha(u_\alpha) + \lambda_\alpha c_\sigma = 0$$

$$\lambda_\alpha(\langle c_\sigma, u_\alpha\rangle - \beta) = 0.$$

Therefore, by virtue of Kuhn-Tucker theorem, we deduce that u_α is the solution of the following extremal problem

$$T_\alpha(u) \to \inf, u \in U_\sigma = \{u \in H : \langle c_\sigma, u\rangle \le \beta\}.$$

4.2. Convergence and rate of convergence.

Theorem 4.1. *Let $c_\sigma, c \in H, f, f_\delta \in F, A, A_\mu \in L(H,F)$, are such that*

$$\|A - A_\mu\| \le \mu,\ \|A_\mu\|^2 \le a,\ \|f - f_\delta\| \le \delta, \|c - c_\sigma\| \le \sigma.$$

Assume that the system of the functions $\{g_\alpha\}$ satisfies the conditions (17)-(19).

If the parameter $\alpha = \alpha(\mu, \delta, \sigma)$ is chosen such that

$$\alpha(\mu,\delta,\sigma) \to 0,\ \ \frac{\mu + \delta^2 + \sigma}{\alpha(\mu,\delta,\sigma)} \to 0\ \ (\mu,\delta,\sigma \to 0)$$

then

$$u_{\alpha(\mu,\delta,\sigma)} \to u_* \ \ (\mu, \delta, \sigma \to 0) \tag{21}$$

If, in addition, the elements u_∞ *and* Pc *can be represented in the form*

$$u_\infty = (A^*A)^p v, \ \ Pc = (A^*A)^{q+1} w, \ \ v, w \in H, 0 < p, q \le p_0 \tag{22}$$

then for

$$\alpha = d(\mu + \delta + \sigma)^{\min\{\frac{1}{p+1}, \frac{1}{q+1}, \frac{1}{2}\}}, d \equiv const > 0 \tag{23}$$

the following inequality is valid

$$\|u_* - u_\alpha\| \le d_p(\mu + \delta + \sigma)^{\min\{\frac{p}{p+1}, \frac{q}{q+1}, \frac{1}{2}\}}. \tag{24}$$

Proof. Suppose that $(I - P)c = 0$ and $c \notin R(A^*A)$. Then $\langle u_\infty, c\rangle \le \beta$, i.e. $u_* = u_\infty$. Let us denote by v_α the solution of the extremal problem $T_\alpha(u) \to \inf, u \in U$. Let us estimate the value $\| v_\alpha - u_\alpha \|$. The element v_α is determined by

$$v_\alpha = g_\alpha(A_\mu^* A_\mu)(A_\mu^* f_\delta - s_\alpha c)$$

where

$$s_\alpha = \begin{cases} 0, & \langle w_\alpha, c\rangle \le \beta \\ \frac{\langle w_\alpha, c\rangle - \beta}{\langle g_\alpha(A_\mu^* A_\mu)c, c\rangle}, & \langle w_\alpha, c\rangle > \beta \end{cases}$$

Hence

$$g_\alpha^{-1}(A_\mu^* A_\mu) v_\alpha - A_\mu^* f_\delta + s_\alpha c = 0 \tag{25}$$

$$s_\alpha(\langle v_\alpha, c\rangle - \beta) = 0 \tag{26}$$

$$g_\alpha^{-1}(A_\mu^* A_\mu) u_\alpha - A_\mu^* f_\delta + \lambda_\alpha c_\sigma = 0 \tag{27}$$

$$\lambda_\alpha(\langle u_\alpha, c_\sigma\langle - \beta) = 0 \tag{28}$$

Using the equalities (16)-(19) and taking into account the properties of the functions g_α we obtain the following inequality

$$\| v_\alpha - u_\alpha \| \le k\frac{\sigma}{\alpha}$$

Multiplying the equalities

$$v_\alpha - u_* = w_\alpha - u_\infty - s_\alpha g_\alpha(A_\mu^* A_\mu)c$$

by $g_\alpha(A_\mu^* A_\mu)^{-1}(v_\alpha - u_*)$ and using again the properties of the functions G_α, we have

$$\beta\alpha \parallel v_\alpha - u_* \parallel^2 \leq \parallel A_\mu(w_\alpha - u_\infty) \parallel^2 + \beta\alpha \parallel w_\alpha - u_\infty \parallel^2 .$$

Combining these inequalities we obtain

$$\parallel u_\alpha - u_* \parallel \leq \parallel u_\alpha - v_\alpha \parallel + \parallel v_\alpha - u_* \parallel \leq$$

$$\frac{1}{\sqrt{\alpha\beta}} \parallel A_\mu(w_\alpha - u_\infty) \parallel + \parallel w_\alpha - u_\infty \parallel + k\frac{\sigma}{\alpha}.$$

It follows from (20) and (21) that $u_{\alpha(\mu,\delta,\sigma}$ tend to u_* when $\mu, \delta, \sigma \to 0$.

Let us consider the case $c \in R(A^*A)$. Then the convergence (22) follows from the equality

$$u_\alpha - u_\infty = w_\alpha - u_\infty + \frac{\langle u_\infty, c\rangle - \beta}{\parallel Ah \parallel^2} h - \frac{\langle w_\alpha, c_\sigma) - \beta}{\langle g_\alpha(A_\mu^* A_\mu) c_\sigma, c_\sigma\rangle} g_\alpha(A_\mu^* A_\mu) c_\sigma,$$

Lemma 3.2. (ii) and (21).

Finally, let $(I - P)c \neq 0$. Then the convergence (12) may be derived as the consequence of the equality

$$u_\alpha - u_\infty = w_\alpha - u_\infty + \frac{\langle u_\infty, c\rangle - \beta}{\|(I-P)c\|^2}(I-P)c -$$

$$\frac{\langle w_\alpha, c_\sigma) - \beta}{\langle \alpha\beta g_\alpha(A_\mu^* A_\mu) c_\sigma, c_\sigma\rangle} \alpha\beta g_\alpha(A_\mu^* A_\mu) c_\sigma,$$

Lemma 3.2. (i) and the relation (21).

It remains to prove the inequality (25), under the additional assumptions (23) and (24). Firstly, we note that in [10] (Theorem 2.4, p. 100), under the condition that $u_\infty = (A^*A)^p v$, the following estimate was proved for $\parallel w_\alpha - u_\infty \parallel$:

$$\|w_\alpha - u_\infty\| \leq d_p(\alpha^p + \frac{\mu + \delta}{\alpha}) \tag{29}$$

If $\langle u_\infty, c\rangle < \beta$, then (21) implies that $\langle w_\alpha, c_\sigma\rangle < \beta$ for small enough μ, δ and σ. Thus, we have that $u_\alpha = w_\alpha$. Hence, the inequality (30) is holds in this case as well.

Let us consider the case $\langle u_\infty, c\rangle \geq \beta$. In the same way as in [10] (Theorem 2.4., p. 100), for $c \in R(A^*A)$ we get

$$\parallel u_\alpha - u_* \parallel \leq d_p(\parallel w_\alpha - u_\infty \parallel + (1 + \ln \mid \mu \mid)\mu^{\min\{1,2q\}} + \alpha^q + \frac{\mu + \sigma}{\alpha}) \tag{30}$$

and

$$\| u_\alpha - u_* \| \leq d_p(\|w_\alpha - u_\infty\| + \alpha + \frac{\mu+\sigma}{\alpha}) \text{ for } (I-P)c \neq 0. \qquad (31)$$

Then, the condition (14) and the inequalities (30)-(32) imply the estimate (25). This completes the proof of the Theorem. □

Theorem 4.2. *If $R(A)$ is a closed subspace of space H, and if*

$$\alpha = d(\mu + \delta + \sigma)^{\frac{1}{2}}, d \equiv const > 0,$$

then

$$\| u_\alpha - u_* \| \leq k(\mu + \delta + \sigma)^{\frac{1}{2}}.$$

At the end, note that Theorem 4.1 gives the potential possibilities of the method. In practice, we do not have the information of type (13) for the properties of the solution u_∞ and the element c. In this case, the choice of the parameter α is not easy at all. In [10], for the operator equations, authors considered so called aposterior choice of the parameter of regularization α. This choice does not include any information about the properties of the solution u_∞. Let us remark that the aposterior choice of the parameter α for the problem (1) is also possible.

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