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Sub-Riemannian geometry, control systems and stochastic flows

Abstract

This article is a survey on some classical and recent results and relations between several mathematical theories. Results from Malliavin stochastic calculus of variations are used to examine known facts from sub-Riemmanian geometry and Geometric control theory from the different point of view. Recent papers establish the bridges between theories that have been considered as far from each other.
Key words: sub-Riemannian geometry, local controllability, Ito diffusions

1. Sub-Riemannian geometry

Let M be a smooth n-dimensional manifold and $A = \sum_{ij} a_{ij}(x)\partial_{x_i}\partial_{x_j}$ semi-elliptic second order operator on $M$with the principal symbol semi-positive smooth quadratic form $a(x,\xi)$. Sub-Riemannian structure is Legendre transform of the quadratic form $a(x,\xi)$ on cotangent bundle T_x^*M :

$$\frac{1}{2}g(x,v) = \sup_{\xi\in T_x^*M(v,\xi)-\frac{1}{2}a(x}, \xi) \tag{1}$$

Therefore, $g = a^*$ and $a = g*$.

Indeed, denote by g_x the restriction of g to T_xM and introduce the linear map $a_x : T_x^*M \mapsto T_xM$ associated with the quadratic form $a(x)$. Then g_x is a positive definite quadratic form on $F_x = Im\,\alpha_x = (Ker\,\alpha_x)^\perp$ (and equal to $+\infty$ outside F_x.) (See, for instance, [4].)

If operator A can be written as: $A = \sum_{i=1}^m V_i^2$, then $a(x,\xi) = \sum_{i=1}^m \langle V_i(x),\xi\rangle^2$ and $g(x,v)$, obtained from $a(x,\xi)$ by Legendre transform is the sub-Riemannian metrics attached to vector fields $V_1,\ldots,V_m$.

"Sums of squares" hypothesis. *For the operator A with the principal symbol a the "sums of squares" hypothesis asserts that A can be written as:*

$$A = \sum_{i=1}^{m} V_i^2 + W\text{"}eqno(2)$$

with smooth $V_1, \ldots, V_m, W$.

In coordinate terms, denoting by $a(x) = (a_{ij}(x))$ the matrix of the quadratic form $a(x,\xi)$ the hypothesis is equivalent to the existence of a $n \times m$ matrix $V(x) = (V_{ij}(x)), i = \overline{l,m}, j = \overline{1,n}$, depending smoothly on x, such that:

$$a(x) = V(x)^T V(x). \qquad (3)$$

Such matrix $V(x)$ does not always exist [11]. However, if the $rank\, a(x)$ is constant, decomposition (3) exists at least locally.

As we can see, sub-Riemannian structures are Legendre transforms of those semi-elliptic operators that can be written as sum of squares.

Recall, that the vector fields $V_1(x), \ldots, V_m(x)$, generating the sub-Riemannian structure satisfy the following so-called Chow condition (or bracket generating condition):

At each point $x \in M$ vector fields $V_1, \ldots, V_m$ together with all possible Lie brackets of any order $[V_i, V_j], [V_i, [V_j, V_k]], \ldots$ span the whole space $T_x^* M$.

Denote by D distribution (or, in general case, differential system), generated by vector fields $V_1, \ldots, V_m$ i.e.

$$D_x = Lin\{V_1(x), \ldots, V_m(x), [V_i, V_j](x), [V_i, [V_j, V_k]](x), \ldots.$$

Absolutely continuous curve $x(t)$ is admissible, if it horizontal to distribution D, i.e. $\dot{x}(t) \in S_{x(t)}$. Famous Chow-Rashevski theorem [5], [13] states that under Chow condition any two points belonging to Mcan be joined by admissible curve. Therefore, under Chow condition, sub-Riemannian structure can be introduced by defining the distance between two points $x, y \in M$ as an infimum length among all admissible curves joining x and y.

Note that in case that $V_1, \ldots, V_m$ are enough to span T^*M we have the Riemannian structure and this is exactly case when operator A is uniformly elliptic.

We can see that in sub-Riemannian case there exists analogue of Hopf-Rinow theorem and it is called Chow-Rashevski theorem. Still, there are essential differences between the two geometries and one of them is existence of singularities in the space of admissible paths joining two fixed points. In other words, shortest geodesies with respect to sub-Riemannian metrics may not be projections of any solution of canonical system of Hamilton equations. Geodesics, that not obey Hamilton equations in differential geometry are called abnormal. Existence of abnormal sub-Riemannian geodesics has been questioned in many papers and only in [10], [12] examples of abnormal geodesics have been demonstrated.

2. Control systems

One of possible means to study sub-Riemannian structure are controlled systems. Namely, we can examine the following system:

$$\dot{x} = \sum_{i=1}^{m} u_i V_i(x),\ x(0) = a, x(1) = b. \tag{4}$$

Consider admissible pair (x^0, u^0) of the system (4). Assume that control u^0 is piecewise smooth at $[0, 1]$. We also may assume that $u^0 \equiv 0$, and obtain the new controlled system:

$$\dot{x} = V_0(t, x) + \sum_{i=1}^{m} u_i V_i(x),\ x(0) = a, x(1) = b. \tag{5}$$

Here, the $V_0(t, x)$ is piecewise smooth vector field, belonging to differential system D and being tangent to trajectory $x^0(t)$. In this fashion the problem of sub-Riemannian geometry can be viewed as the problem of minimization of integral:

$$\int_0^1 \phi(x, u) dt$$

at the trajectories of the system (5).

Introduce the hamiltonian of the system (5):

$$H(t, x, u, \psi, \lambda^0) = \lambda^0 \phi(x, u) + \langle \psi, V_0(t, x) + \sum_{i=1}^{m} u_i V_i(x) \rangle.$$

First order necessary optimality conditions state that if admissible process is optimal the following condition holds:

$$\frac{\partial H}{\partial u}(t, x^0(t), u^0(t), \psi(t), \lambda^0) = 0, \dot{\forall} t, \tag{6}$$

for some $\lambda^0 \in \{0,1\}$ and $n-$dimensional vector function $\psi(t)$, being the solution to adjoint equation:

$$\dot{\psi}(t) = -\frac{\partial H}{\partial u}(t, x^0(t), u^0(t), \psi(t), \lambda^0) = 0,\ \psi(0) = y_1, \psi(1) = -y_2. \qquad (7)$$

Denote by Φ the fundamental matrix of first variation system for (5), i.e. solution of homogenous system:

$$\frac{d}{dt}\Phi = \Phi\left(\frac{\partial V_0(t, x^0(t))}{\partial x} + \sum_{i=1}^{m} u_i^0 \frac{\partial V_i(x^0(t))}{\partial x}\right),\ \Phi(0) = I, \qquad (8)$$

where I is identity matrix. Introduce the controllability matrix:

$$Z = \Phi(1)^T \int_0^1 \Phi^{-1}(t)^T V(x^0)^T V(x^0)^T \Phi^{-1}(t) dt \Phi^{-1}(1).$$

The condition $dim\, Ker\, Z = 0$ is equivalent to controllability of first variation system for (5) and normality of extremal process (x^0, u^0).

In control theory the following theorem is known (see [5]):

Theorem 1. *Assume that abnormal extremal (x^0, u^0) of the system (5) is optimal. Then:*

$$\frac{\partial}{\partial u_i}\frac{d}{dt}\frac{\partial H}{\partial u}(t, x^0(t), u^0(t), \psi(t), \lambda^0) = 0 \qquad (9)$$

where $\psi(t)$ is again the solution of (7).

Reformulated in Lie brackets terms, the equation (9), together with the first order necessary optimality conditions, implies that vector fields $V_i(x); i = \overline{1,m}$ together with their first order Lie brackets (i.e. brackets of the form $[V_i, V_j]$, $1 \le i, j \le m$) do not form rank n system at any point x. Therefore, the Goh condition (together with the first order optimality conditions) turns out to be reformulation of the known in Differential Geometry statement [7], [10] that abnormal geodesics lay entirely in the set of points x where the distribution D fails to be strong bracket generating.

The distribution D is said to be strong bracket generating at x if for any $i = 1$ $dots, m$ the space $Lin\{V_1(x), \ldots, V_m(x), [V_1; V_i](x), \ldots [Vm, V_i](x)\}$ is the whole tangent space at x. In order words, the distribution is strong bracket generating, if vector fields together with their first order Lie brackets alone are enough to span the whole tangent space. In particular, the distribution, corresponding to contact structure [1] is strong bracket generating .

3. Stochastic flows

Consider the vector stochastic differential equation in Stratonovich form:

$$X_t^{x_0} = x_0 + \int_0^t V_0(X_t^{x_0})ds + \sum_{i=1}^m \int_0^t V_i(X_t^{x_0}) \circ dW_s,\ t \in [0,1], \tag{10}$$

where $x_0 \in R^n$, W_t is a $m-$dimensional Brownian motion, $V(x) = [V_{ij}(x)]$ is a $n \times m$ matrix and $V_0(x)$ is vector in R^n with components $V_0^k(x)$. We assume that for all $i = 1, \ldots, n$, $j = 0, \ldots, m$ the functions V_{ij} are sufficiently smooth.

Consider the infinitesimal generator of (10):

$$Au = \sum_{i=1}^n \frac{\partial}{\partial x_i}\left(\sum_{j=1}^n a_{ij}(x)\frac{\partial u}{\partial x_j}\right) - \sum_{i=1}^n \frac{\partial (V_0^i(x)u)}{\partial x_i}, \tag{11}$$

where $a_{ij}(x) = \frac{1}{2}\sum_{k=1}^m V_k^i(x)V_k^j(x)$.

Definition 1. *A differential operator A defined on an open set $\mathcal{O} \subset R^n$ is called hypoelliptic if, whenever u is a distribution on $\mathcal{O}$, u is a smooth function on any open set $\tilde{\mathcal{O}} \subset \mathcal{O}$ on which Au is smooth.*

In PDE theory the Chow condition is known as Hoörmander condition. Namely:

Definition 2. *We will say that differential system $V_1, \ldots, V_m$ satisfies the Hoörmander condition, if for every $x \in R^n$ vectors*

$$V_1(x)\ldots, V_m(x), [Vi, V_j](x)_{0\le i,j\le m}, [V_i, [V_j, V_k]](x)_{0\le i,j,k\le m}, \ldots,$$

span the space R^n.

Theorem 2. (Hoörmander [8])*Assume that the system $V_1 \ldots, V_m$ satisfies Hoörmander condition. Then the operator $A = \sum_{i=1}^m V_i^2$ is hypoelliptic.*

Note that this is also necessary condition if V_i's are with analytic coefficients. Still, as shown in [11] it is possible to obtain hypoellipticity also for second order differential operators that can not be written as sums of squares.

The Hoörmander theorem has been proven in [Malliavin] by using probabilistic tools. In order to reprove the Hoörmander theorem Malliavin introduced covariance matrix:

$$\Gamma = J_{0\to 1}^T \int_0^1 J_{0\to 1}^{-1T} V(X_t)^T J_{0\to 1}^{-1} dt J_{0\to 1}, \tag{12}$$

where $(J_{0\to t})_{t\geq 0}$ is the first variation process defined by:

$$J_{0\to t} = \partial X_t^x \partial x,$$

and V denotes the $n \times m$ matrix $(V_1, \ldots, V_m)$.

In [9] it is shown that under H0rmander condition the covariance matrix Γ is invertible almost surely, which guarantees existence of smooth transition densities for the random variable $X_1^{x_0}$ and hypoellipticity of operator A.

The special case of Hörmander theorem, when $V_1, \ldots, V_m$ are alone enough to span R^n has been obtained by Hermann Weyl. This is exactly the case when A is elliptic, which leads to Riemannian geometry.

4. Conclusion

From [11] we can see that sub-Riemannian geometry is not exactly associated with all hypoelliptic operators. Instead, the sub-Riemannian geometry is naturally associated with Ito diffusions described by stochastic differential equations od the type (10). The infinitesimal generators of Ito diffusions are exactly hypoelliptic that can be written as sums of squares. Sub-Riemannian structures arise as Legendre transforms of these operators. For more details see [3]. The gap between Hörmander condition and hypoellipticity is precisely described in probabilistic terms in [2].

Abnormal paths are investigated in sub-Riemannian geometry and Geometric control theory since the system can examine various interesting phenomena in their neighborhood. In [10] it is shown that such abnormal path can be "rigid" in the sense it is isolated point in the space of all admissible paths in C1 metrics. However, as proven in [9] under Hormander condition the covariance matrix is invertible almost surely, which implies that rigid paths in sub-Riemannian geometry are almost impossible (in the sense of probability).

While abnormal paths may appear in sub-Riemannian geometry, they do not exist in at least two cases: elliptic case (when the vector fields alone are enough to span the whole space - Riemannian geometry) and the strong bracket generating case (when vector fields together with their first order Lie brackets only are enough to span the whole space - contact geometry).

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