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GENERALIZED GRADIENT METHODS FOR QUASI-VARIATIONAL INEQUALITIES

Abstract

In this paper we study generalized gradient-type methods for solving quasi-variational inequities. We establish sufficient conditions for the convergence of the proposed methods and estimate the rates of convergence.

Key words: gradient-type method, continuous methods, iterative methods, quasi-variational inequlities

GENERALIZOVANE GRADIJENTNE METODE ZA KVAZI-VARIJACIONE NEJEDNAKOSTI

U radu su predstavljene generalizovane metode gradijentnog tipa za rješavanje kvazi-varijacionih nejednakosti. Izvedeni su uslovi konvergencije predstavljenih metoda i dobijene ocjene brzine konvergencije.

Ključne riječi: metode gradijentnog tipa, neprekidne metode, iterativne metode, kvazi-varijacione nejednakosti

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1. INTRODUCTION

In this paper, the object of our investigation is the following quasi-variational inequality:

find $x^* \in C(x^*)$ such that

$$\langle F(x^*), y - x^* \rangle \geq 0, \quad \forall y \in C(x^*). \tag{1.1}$$

Here $F : H \to H$ is a continuous mapping and $C : H \to 2^H$ is a set-valued mapping such that $C(x)$ is a nonempty closed convex subset of Hilbert space H for each $x \in H$. Let us note that $\langle \cdot, \cdot \rangle$ denotes the inner product in H and $\| \cdot \|$ – corresponding norm.

A particularly well known and studied case occurs when $C(x)$ is independent of x, so that, $C(x) \equiv C$, for all x, where X is nonempty closed convex set. In this case quasi-variational inequality (1.1) becomes classical variational inequality of Stampacchia type (see [14]) which consists of finding $x^* \in C$ such that

$$\langle F(x^*), y - x^* \rangle \geq 0, \quad \forall y \in C. \tag{1.2}$$

The theory and methods for solving variational inequalities are thoroughly treated in the scientific literature. Quasi-variational inequalities were introduced by Bensoussan et al. (see [6,7]) in connection with the study of the impulse control problems. A thorough study of these probems can be found in [5,20]. In the last decades the theory of quasi-variational inequalities attracts considerable interest of scientists. This theory develops powerful mathematical models which unify important concepts in applied mathematics, like systems of nonlinear equations, optimality conditions for optimization problems, game and equilibrium problems (see, for instance [2-4,8,10,13,15,17-19,21-26]).

The existence and approximation theories for quasi-variational inequalities require that a variational inequality and a fixed point problem should be solved simultaneously. Consequently, many solution techniques for variational inequalities have not been adapted for quasi-variational inequalities, and there are many questions to be answered.

The paper is organized as follows. In the second section, we introduce the problem of quasi-variational inequality and recall the main known results that will be used in the next sections. In the third section, we present some known variants of the projection gradient-type methods and their generalizations. In the last section, iterative and continuous generalizations of projection gradient-type method are proposed. In the same section, we also present a convergence analysis of these methods.

2. PRELIMINARIES

We start this section by recalling some notions and results which will be useful within this paper.

Definition. Let C be a nonepmty susbset of the real Hilbert space H. The mapping $F : H \to H$ is said to be

(a) monotone on C, if for every $x, y \in C$ it holds

$$\langle F(y) - F(x), y - x \rangle \geq 0;$$

(b) pseudo-monotone on C, if for every $x, y \in C$ it holds

$$\langle F(x), y - x \rangle \geq 0 \Rightarrow \langle F(y), y - x \rangle \geq 0;$$

(c) strongly monotone on C with $\mu > 0$, if for every $x, y \in C$ it holds

$$\langle F(y) - F(x), y - x \rangle \geq \mu \|x - y\|^2;$$

(d) strongly pseudo-monotone on C with $\mu > 0$, if for every $x, y \in C$ it holds

$$\langle F(x), y - x \rangle \geq 0 \Rightarrow \langle F(y), y - x \rangle \geq \mu \|x - y\|^2;$$

(e) Lipschitz continuous with Lipschitz constant $L > 0$, if for every $x, y \in C$ it holds

$$\|F(y) - F(x)\| \leq L \|y - x\|.$$

The constant $\mu \geq 0$ from (c) (or (d)) is a parameter of strong monotonicity (strong pseudo-monotonicity) of operator F, and L from (a) is a parameter of Lipschitz continuity. If $\mu = 0$, then F is a monotone (pseudo-monotone) operator. From the definition it is clear that $\mu \leq L$.

Let us note that for each point $x \in H$, there exists a unique point in C, denoted by $\mathcal{P}_C(x)$, such that

$$\|x - \mathcal{P}_C(x)\| \leq \|x - y\| \quad \forall y \in C.$$

The mapping $\mathcal{P}_C : H \to H$ is called projection of H onto C and is characterized by the following two properties:

$$\mathcal{P}_C(x) \in C$$

and

$$\langle x - \mathcal{P}_C(x), \mathcal{P}_C(x) - y\rangle \geq 0 \quad \forall y \in C, \tag{2.1}$$

and if C is a hyperplane, then (2.1) becomes an equality. It is known that $\mathcal{P}_C(x)$ is a singleton set if C is nonempty closed and convex. In this case, $\mathcal{P}_C$ is Lipschitz continuous with constant $L = 1$.

In what follows, we will use known fixed point formulation for the solution of quasi-variational inequality: $x_* \in C(x_*)$ is a solution of problem (1.1) if and only if for any $\alpha > 0$ it holds that

$$x_* = \mathcal{P}_{C(x_*)}[x_* - \alpha F(x_*)]. \tag{2.2}$$

The following result gives sufficient conditions for the existence of unique solution of quasi-variational inequality (1.1).

Theorem 1. [24] *If the map F is Lipschitz continuous and strongly monotone on H with constants L and $\mu > 0$, respectively, and C is a set-valued mapping with nonempty closed and convex values such that*

$$\|\mathcal{P}_{C(x)}(z) - \mathcal{P}_{C(y)}(z)\| \leq l\|x - y\|, \quad l + \sqrt{1 - \mu^2/L^2} < 1, \ \ \forall\, x, y, z \in H. \tag{2.3}$$

then the problem (1.1) has a unique solution.

Nesterov and Scrimali (see [21]) proved that in (2.3) is sufficient to require $l < \frac{\mu}{L}$.

Let us mention that assumption (2.3) is a kind of strengthening of the contraction property for multifunction $C(x)$. An example of such mapping is given in the following theorem:

Theorem 2. [21] *Let function $c : H \to H$ be Lipschitz continuous with Lipschitz constant l and set C_0 be a closed convex set. Then*

$$C(x) := c(x) + C_0$$

satisfies (2.3) with the same value of l.

This case of quasi-variational inequalities is most often discussed in the literature and it is known as the moving set.

3. PROJECTION GRADIENT METHODS AND GENERALIZATIONS

In this section, we present iterative and continuous projection gradient methods and its generalizations.

Using the fixed point formulation (2.2), the following method for solving problem (1.1) is used.

Algorithm 3.1
Data: $x_0 \in H$ and $\alpha > 0$.
Step 0: Set $k = 0$.
Step 1: If $x_k = \mathcal{P}_{C(x_k)}[x_k - \alpha F(x_k)]$ stop.
Step 2: Set $x_{k+1} = \mathcal{P}_{C(x_k)}[x_k - \alpha F(x_k)]$ and $k \leftarrow k + 1$; go to Step 1.

Nesterov and Scrimali have proved convergence of this algorithm, (see [21]). Consequently, for a given quasi-variational inequality with a strong monotone and Lipschitz continuous mapping F, provided that the constants L and μ are available, and for optimal stepsize $\alpha = \frac{\mu}{L^2}$, the sequence (x_k) defined iteratively by

$$x_{k+1} = \mathcal{P}_{C(x_k)}[x_k - \alpha F(x_k)] \tag{3.1}$$

converges to the unique solution of the quasi-variational inequality, for any x_0 in H.

One can apply similar analysis to a variable-step projection scheme in which the step size α is allowed to vary from one iteration to the next. This extends iteration (3.1) which is a constant-step projection scheme. The resulting variable-step algorithm is not a line-search method, however, because the step size α_k is not determined by a line search routine.

Algorithm 3.2
Data: $x_0 \in H$.
Step 0: Set $k = 0$.
Step 1: Choose $\alpha_k > 0$. Set $x_{k+1} = \mathcal{P}_{C(x_k)}[x_k - \alpha_k F(x_k)]$.
Step 2: If $x_{k+1} = x_k$ stop.
Step 3: Set $k \leftarrow k + 1$; go to Step 1.

The choice of the sequence of scalars (α_k) may be crucial for the success of Algorithm 3.2.

It is interesting to construct a continuous variant of this method. For given $x_0 \in H$, we construct trajectory that start at an arbitrary point $x_0 \in H$, and during the time converge to the solution of (1.1):

$$x'(t) + x(t) = \mathcal{P}_{C(x(t))}[x(t) - \alpha F(x(t))], \quad x(0) = x_0.$$

The proof of convergence was exposed in [13].

To speed up the convergence, we can use more generalized variant of the previous method:

$$x'(t) = -a(t)x(t) + a(t)\mathcal{P}_{C(x(t))}[x(t) - \alpha(t)F(x(t))], \quad x(0) = x_0. \tag{3.2}$$

Here $a(t) > 0$ and $\alpha(t) > 0$ are parameters of the method. The existence and uniqueness of the trajectory $x \in C^1([0, +\infty), H)$ generated by (3.2) has been established as a consequence of the global CauchyпїSLipschitz Theorem and by making use of the Lipschitz continuity of F (see [11]).

In this paper we prove the convergence of the trajectory generated by (3.2) in the case of moving set.

Iterative variant of proposed continuous method (3.2) has a form:

Algorithm 3.3
Data: $x_0 \in H$.
Step 0: Set $k = 0$.
Step 1: Choose $0 < h_k \leq 1$. Compute

$$x_{k+1} = (1 - h_k)x_k + h_k \mathcal{P}_{C(x_k)}[x_k - \alpha_k F(x_k)], \tag{3.3}$$

Step 2: If $x_{k+1} = x_k$ stop.
Step 3: Set $k \leftarrow k + 1$ and go to Step 1.

Parameters $0 < h_k \leq 1$ and $\alpha_k > 0$ can be chosen on different ways. In the next paragraph, we prove the convergence of this method in the case of moving set.

Let us remark that fixed point formulation (2.2) of problem (1.1) can be written in the form

$$\begin{aligned} u &= \mathcal{P}_{C(x)}[x - \alpha F(x)], \\ x &= \mathcal{P}_{C(u)}[u - \alpha F(u)]. \end{aligned}$$

This formulation enables to suggest and analyze the following two-step method for solving quasi-variational inequality (1.1).

Algorithm 3.4
Data: $x_0 \in H$ and $\alpha > 0$.
Step 0: Set $k = 0$.
Step 1: If $x_k = \mathcal{P}_{C(x_k)}[x_k - \alpha F(x_k)]$ stop.
Step 2: Choose $a_k > 0$ and $b_k > 0$. Compute

$$u_k = (1 - b_k)x_k + b_k \mathcal{P}_{C(x_k)}[x_k - \alpha F(x_k)], \tag{3.4}$$

$$x_{k+1} = (1 - a_k)x_k + a_k \mathcal{P}_{C(u_k)}[u_k - \alpha F(u_k)], \tag{3.5}$$

set $k \leftarrow k + 1$ and go to Step 1.

Here, $0 < a_k \leq 1$, $0 \leq b_k \leq 1$ for all $k \geq 0$ and $\alpha > 0$ are parameters of method.

In the case of $b_k \equiv 0$, algorithm 3.4 turn into algorithm 3.3. This method was suggested by Mijajlović at al. in 2018 (see [18]).

Shehu et al. (see [26]) have proposed a generalized gradient-type method with intertial extrapolation step:

Algorithm 3.5
Data: Select arbitrary starting points $x_0, x_1 \in H$.
Step 0: Set $k = 1$.
Step 1: Choose $\theta_k > 0$ and $a_k > 0$. Set

$$y_k = x_k + \theta_k(x_k - x_{k-1})$$

$$x_{k+1} = (1 - a_k)y_k + a_k\mathcal{P}_{C(y_k)}[y_k - \alpha F(y_k)].$$

Step 2: If $x_{k+1} = x_k$ stop.
Step 3: Set $k \leftarrow k + 1$ and go to Step 1.

Here (θ_k) and (a_k) are sequences satisfying different sets of conditions. Authors have proved the strong convergence of Algorithm 3.5 in the particular case.

Korpelevich (see [16]) combined two neighboring iterations to provide a new projection method for finding saddle points which was called the extragradient method. Antipin at al. (see [4]) have proved the convergence of extragradient method for solving quasi-variational inequalities:

Algorithm 3.6
Data: $x_0 \in H$ and $\alpha > 0$.
Step 0: Set $k = 0$.
Step 1: If $x_k = \mathcal{P}_{C(x_k)}[x_k - \alpha F(x_k)]$ stop.
Step 2: Compute

$$\begin{aligned} \bar{x}_k &= \mathcal{P}_{C(x_k)}[x_k - \alpha F(x_k)], \\ x_{k+1} &= \mathcal{P}_{C(x_k)}[x_k - \alpha F(\bar{x}_k)], \end{aligned}$$

set $k \leftarrow k + 1$ and go to Step 1.

4. MAIN RESULTS

Generalized continuous method (3.2) in case of moving set is described by the following dynamical system

$$\frac{x'(t)}{a(t)}+x(t) = c(x(t))+\mathcal{P}_{C_0}[x(t)-c(x(t))-\alpha(t)F(x(t))],\ t \geq 0, \quad (4.1)$$

where $x(0) = x_0$ is initial point and $0 < a(t) \leq 1$, $\alpha(t) > 0$ are parameters of the method.

In what follows we will investigate the asymptotic behavior of the trajectory generated by dynamical system (4.1).

Theorem 3. *Let H be a Hilbert space, $C_0 \subset H$ be nonempty, closed, and convex, $c : H \to H$ be $l-$Lipschitz continuous, and $C : H \to 2^H$ be a set-valued map such that $C(x) := c(x) + C_0$, for every $x \in H$. Assume that for $\mu > 0$ and $L > 0$, the map $F : H \to H$ satisfies the following condition:*

$$\|F(u)-F(v)\|^2+\mu L\|u-v\|^2 \leq (L+\mu)\langle F(u)-F(v), u-v\rangle, \quad \forall u, v \in H. \quad (4.2)$$

Parameters $a(t) \in C([0,+\infty))$ and $\alpha(t) > 0$ satisfy

$$l < \frac{2\mu}{3L}, \quad 2L^2 + 4\mu^2L^2 - 1 > 0, \quad \alpha_1 \leq \alpha(t) \leq \alpha_2,$$

$$\alpha_1 = 4(\mu - lL) - 4A,$$

$$\alpha_2 = \min\left\{4\mu - 4lL + 4A, \frac{8\mu^2}{L+\mu}\right\}$$

$$\int_0^\infty a(t)dt = +\infty,$$

where $A = \sqrt{(\mu - lL)^2 - \frac{1}{4}l^2L^2}$.

Then, for every initial approximation $x_0 \in H$, trajectory $x(t)$ generated by (4.1) converges to the unique solution $x_* \in C(x_*)$ of problem (1.1) with the following rate:

$$\|x(t) - x_*\| \leq \exp\left\{-A^2 \int_0^t a(\xi)d\xi\right\} \|x_0 - x_*\|.$$

Proof. First of all, note that condition (4.2) implies that F is $\mu-$strongly monotone and $L-$Lipschitz continuous (see [27], page 211). Indeed, setting $w = \|F(u) - F(v)\|$ in (4.2), we have

$$w^2 - (L+\mu)\|u-v\|w + L\mu\|u-v\|^2 \leq 0.$$

Then, this inequality is satisfied for

$$w_1 \leq w \leq w_2,$$

where $w_1 = \mu\|u-v\|$ and $w_2 = L\|u-v\|$, i.e.

$$\mu\|u-v\| \leq \|F(u) - F(v)\| \leq L\|u-v\|. \tag{4.3}$$

It follows form (4.3) that F is Lipschitz continuous with the constant L. From (4.2) and (4.3), we get

$$\begin{aligned}\mu(\mu+L)\|u-v\|^2 &= \mu^2\|u-v\|^2 + \mu L\|u-v\|^2 \\ &\leq \|F(u)-F(v)\|^2 + \mu L\|u-v\|^2 \\ &\leq (L+\mu)\langle F(u)-F(v), u-v\rangle,\end{aligned}$$

confirming that F is strongly monotone with the constant μ.

Hence, conditions from Theorem 1 are satisfied, so the unique solution $x_* \in C(x_*)$ of problem (1.1) exists.

From the definition of the trajectory $x(t)$, we conclude that $\frac{x'(t)}{a(t)} + x(t) - c(x(t))$ belongs to the set C_0. Having in mind that x_* is a solution of quasi-variational inequality, we have that (1.1) is valid for $y = \frac{x'(t)}{a(t)} + x(t) - c(x(t)) + c(x_*) \in C(x_*)$:

$$\alpha(t)\left\langle F(x_*), \frac{x'(t)}{a(t)} + x(t) - c(x(t)) + c(x_*) - x_*\right\rangle \geq 0, \tag{4.4}$$

for all $\alpha(t) > 0$.

Due to the variational characterization of the projection in (4.1), we have

$$\left\langle \frac{x'(t)}{a(t)} + \alpha(t)F(x(t)), z - \frac{x'(t)}{a(t)} - x(t) + c(x(t))\right\rangle \geq 0, \quad t \geq 0,$$

which is valid for all $z \in C_0$. For the choice $z = x_* - c(x_*) \in C_0$, it implies

$$\left\langle \frac{x'(t)}{a(t)} + \alpha(t)F(x(t)), x_* - \frac{x'(t)}{a(t)} - x(t) + c(x(t)) - c(x_*) \right\rangle \geq 0, \ \ t \geq 0. \tag{4.5}$$

Combining (4.4) and (4.5), we obtain

$$\begin{aligned} &\left\langle \frac{x'(t)}{a(t)}, \frac{x'(t)}{a(t)} + x(t) + c(x_*) - c(x(t)) - x_* \right\rangle \\ \leq\ & \alpha(t) \left\langle F(x(t)) - F(x_*), x_* - \frac{x'(t)}{a(t)} - x(t) + c(x(t) - c(x_*)) \right\rangle, \end{aligned}$$

and then rearranging the above inequality as follows, we get

$$\begin{aligned} &\left\langle \frac{x'(t)}{a(t)}, \frac{x'(t)}{a(t)} \right\rangle + \left\langle \frac{x'(t)}{a(t)}, x(t) - x_* \right\rangle + \left\langle \frac{x'(t)}{a(t)}, c(x_*) - c(x(t)) \right\rangle \\ \leq\ & \alpha(t) \left\langle F(x(t)) - F(x_*), -\frac{x'(t)}{a(t)} \right\rangle \\ & + \alpha(t)\langle F(x(t)) - F(x_*), x_* - x(t)\rangle \\ & + \alpha(t)\langle F(x(t)) - F(x_*), c(x(t) - c(x_*))\rangle, \quad t \geq 0. \end{aligned} \tag{4.6}$$

We will now estimate the terms appearing in (4.6) as follows:

$$\left\langle \frac{x'(t)}{a(t)}, \frac{x'(t)}{a(t)} \right\rangle = \frac{1}{a^2(t)}\|x'(t)\|^2,$$

and

$$\left\langle \frac{x'(t)}{a(t)}, x(t) - x_* \right\rangle = \frac{1}{2a(t)} \frac{d}{dt}\|x(t) - x_*\|^2.$$

Using Young inequality and $l-$Lipschitz continuity of c, we obtain

$$\begin{aligned} &\left\langle \frac{x'(t)}{a(t)}, c(x_*) - c(x(t)) \right\rangle \geq -\frac{1}{a(t)}\|x'(t)\| \cdot \|c(x_*) - c(x(t))\| \\ \geq\ & -\frac{1}{2a^2(t)L^2}\|x'(t)\|^2 - \frac{L^2}{2}\|c(x_*) - c(x(t))\|^2 \\ \geq\ & -\frac{1}{2a^2(t)L^2}\|x'(t)\|^2 - \frac{l^2L^2}{2}\|x(t) - x_*\|^2, \end{aligned}$$

$$\alpha(t)\left\langle F(x(t))-F(x_*),-\frac{x'(t)}{a(t)}\right\rangle \leq \frac{\alpha^2(t)}{8\mu^2}\|F(x(t))-F(x_*)\|^2 +\frac{2\mu^2}{a^2(t)}\|x'(t)\|^2,$$

$$\alpha(t)\left\langle F(x(t))-F(x_*),c(x(t))-c(x_*)\right\rangle \leq \alpha(t)Ll\|x(t)-x_*\|^2.$$

By substituting the above two equalities and three inequalities into (4.6), we obtain

$$\begin{aligned}&\frac{1}{a^2(t)}\|x'(t)\|^2+\frac{1}{2a(t)}\frac{d}{dt}\|x(t)-x_*\|^2-\frac{1}{2a^2(t)L^2}\|x'(t)\|^2\\ &-\frac{l^2L^2}{2}\|x(t)-x_*\|^2 \leq \frac{\alpha^2(t)}{8\mu^2}\|F(x(t))-F(x_*)\|^2\\ &+\frac{2\mu^2}{a^2(t)}\|x'(t)\|^2+\alpha(t)Ll\|x(t)-x_*\|^2\\ &+\alpha(t)\left\langle F(x(t))-F(x_*),x_*-x(t)\right\rangle,\end{aligned}$$

which implies that

$$\begin{aligned}&\frac{1}{2L^2a^2(t)}\left(2L^2+4\mu^2L^2-1\right)\|x'(t)\|^2+\frac{1}{2a(t)}\frac{d}{dt}\|x(t)-x_*\|^2\\ \leq\ &\left(\alpha(t)lL+\frac{l^2L^2}{2}\right)\|x(t)-x_*\|^2+\frac{\alpha^2(t)}{8\mu^2}\|F(x(t))-F(x_*)\|^2\\ &+\alpha(t)\left\langle F(x(t))-F(x_*),x_*-x(t)\right\rangle, \qquad (4.7)\end{aligned}$$

Setting $u=x(t)$ and $v=x_*$ into (4.2) leads to the following inequality

$$\|F(x(t))-F(x_*)\|^2 \leq (L+\mu)\langle F(x(t))-F(x_*),x(t)-x_*\rangle -L\mu\|x(t)-x_*\|^2$$

and, as a result, we deduce

$$\frac{1}{2L^2a^2(t)}\left(2L^2+4\mu^2L^2-1\right)\|x'(t)\|^2+\frac{1}{2a(t)}\frac{d}{dt}\|x(t)-x_*\|^2$$

$$\leq \left(\alpha(t)lL + \frac{l^2L^2}{2} - \frac{\alpha^2(t)L}{8\mu}\right)\|x(t) - x_*\|^2$$
$$+\left(\frac{\alpha^2(t)}{8\mu^2}(L+\mu) - \alpha(t)\right)\langle F(x(t)) - F(x_*), x(t) - x_*\rangle$$

Taking into account conditions of the theorem, we conclude that $\alpha(t) < \frac{8\mu^2}{L+\mu}$, and with condition of strong monotonocity of F, we get

$$\frac{1}{2L^2a^2(t)}\left(2L^2 + 4\mu^2L^2 - 1\right)\|x'(t)\|^2 + \frac{1}{2a(t)}\frac{d}{dt}\|x(t) - x_*\|^2$$
$$\leq \left(\alpha(t)lL + \frac{l^2L^2}{2} + \frac{\alpha^2(t)}{8} - \alpha(t)\mu\right)\|x(t) - x_*\|^2$$

Finally, we have

$$\frac{d}{dt}\|x(t) - x_*\|^2 \leq 2a(t)\left(\alpha(t)lL + \frac{l^2L^2}{2} + \frac{\alpha^2(t)}{8} - \alpha(t)\mu\right)\|x(t) - x_*\|^2.$$

It turns out that the function $B(\alpha) = \alpha lL + \frac{l^2L^2}{2} + \frac{\alpha^2}{8} - \alpha\mu$ is negative on the set $\alpha_1 \leq \alpha(t) \leq \alpha_2$ and attains the minimum value on this set at the point

$$\bar{\alpha} = 4(\mu - lL)$$

and the optimal value is

$$B(\bar{\alpha}) = -2\left(\mu - \frac{3}{2}lL\right)\left(\mu - \frac{1}{2}lL\right) = -2A^2 < 0.$$

From here, taking into account $x(0) = x_0$, we obtain

$$\|x(t) - x_*\| \leq \exp\left\{-A^2\int_0^t a(\xi)d\xi\right\}\|x_0 - x_*\|,$$

Since $\int_0^\infty a(\xi)d\xi = +\infty$ we have that trajectory $x(t)$ generated by (4.1) converges to the unique solution x_*, which completes the proof. $\boxdot$

Now, we will describe an iterative variant of the proposed continuous method.

Method (3.3) from Algorithm 3.3 can be written as

$$\frac{x_{k+1}-x_k}{h_k}+x_k=\mathcal{P}_{C(x_k)}(x_k-\alpha F(x_k)),$$

i.e. in the case of the moving set we have

$$\frac{x_{k+1}-x_k}{h_k}+x_k-c(x_k)=\mathcal{P}_{C_0}(x_k-c(x_k)-\alpha_k F(x_k)). \tag{4.8}$$

We will prove the convergence of this method under new conditions, which offer wider possibilities of the choice of parameters of the method.

Assume that there exists a unique solution of the quasi-variational inequality (1.1), and denote it with $x_*\in C(x_*)=c(x_*)+C_0$. From the definition of the sequence (x_k), we have that each $\frac{x_{k+1}-x_k}{h_k}+x_k-c(x_k)$ belongs to the set C_0. Having in mind that x_* is a solution of the given quasi-variational inequality, we have that is valid for $y=\frac{x_{k+1}-x_k}{h_k}+x_k-c(x_k)+c(x_*)\in c(x_*)+C_0=C(x_*)$ ::

$$\left\langle F(x_*),\frac{x_{k+1}-x_k}{h_k}+x_k-c(x_k)-x_*+c(x_*)\right\rangle\geq 0. \tag{4.9}$$

From (4.8) we conclude that $\frac{x_{k+1}-x_k}{h_k}+x_k-c(x_k)$ is a projection of a certain element on the set C_0. Consequentely, we can write (4.8) as a variational inequality

$$\left\langle\frac{x_{k+1}-x_k}{h_k}+\alpha_k F(x_k),z-\frac{x_{k+1}-x_k}{h_k}-x_k+c(x_k)\right\rangle\geq 0,$$

which is valid for all $z\in C_0$. So, it is valid for $z=x_*-c(x_*)\in C_0$ and we get

$$\left\langle\frac{x_{k+1}-x_k}{h_k}+\alpha_k F(x_k),x_*-c(x_*)-\frac{x_{k+1}-x_k}{h_k}-x_k+c(x_k)\right\rangle\geq 0. \tag{4.10}$$

Multiplying (4.9) by α_k and adding to (4.10), we obtain

$$\left\|\frac{x_{k+1}-x_k}{h_k}\right\|^2+\left\langle\frac{x_{k+1}-x_k}{h_k},x_k-x_*\right\rangle$$

$$
\begin{aligned}
&+\left\langle \frac{x_{k+1}-x_k}{h_k}, c(x_*)-c(x_k)\right\rangle \\
\leq\ & \alpha_k \langle F(x_k)-F(x_*), x_*-x_k\rangle + \alpha_k \langle F(x_k)-F(x_*), c(x_k)-c(x_*)\rangle \\
&+\alpha_k \left\langle F(x_k)-F(x_*), -\frac{x_{k+1}-x_k}{h_k}\right\rangle . \qquad (4.11)
\end{aligned}
$$

Now, we will estimate the terms from this inequality:

$$
\begin{aligned}
\left\langle \frac{x_{k+1}-x_k}{h_k}, x_k-x_*\right\rangle &= h_k \left\langle \frac{x_{k+1}-x_k}{h_k}, \frac{x_k-x_*}{h_k}\right\rangle \\
&= \frac{h_k}{2}\left(\left\|\frac{x_{k+1}-x_*}{h_k}\right\|^2 - \left\|\frac{x_k-x_*}{h_k}\right\|^2 - \left\|\frac{x_{k+1}-x_k}{h_k}\right\|^2\right) \qquad (4.12)
\end{aligned}
$$

Assuming that c is $l-$Lipschitz continuous, we get

$$
\left\langle \frac{x_{k+1}-x_k}{h_k}, c(x_*)-c(x_k)\right\rangle \geq -\frac{1}{4}\left\|\frac{x_{k+1}-x_k}{h_k}\right\|^2 - l^2\left\|x_k-x_*\right\|^2 . \qquad (4.13)
$$

We will also assume that F satisfies (4.2). So, F is $\mu-$strongly monotone and $L-$Lipschitz continuous and we have

$$
\alpha_k \langle F(x_k)-F(x_*), c(x_k)-c(x_*)\rangle \leq \frac{\alpha_k^2}{2}\|F(x_k)-F(x_*)\|^2 + \frac{l^2}{2}\|x_k-x_*\|^2 \qquad (4.14)
$$

and

$$
\begin{aligned}
&\alpha_k \left\langle F(x_k)-F(x_*), \frac{x_k-x_{k+1}}{h_k}\right\rangle \\
\leq\ & \alpha_k^2\|F(x_k)-F(x_*)\|^2 + \frac{1}{4}\left\|\frac{x_{k+1}-x_k}{h_k}\right\|^2 . \qquad (4.15)
\end{aligned}
$$

Putting (4.12),(4.13),(4.14) and (4.15) in (4.11), we obtain

$$
\begin{aligned}
&(1-h_k)\left\|\frac{x_{k+1}-x_k}{h_k}\right\|^2 + \frac{1}{h_k}\|x_{k+1}-x_*\|^2 \\
\leq\ & \left(\frac{1}{h_k}+3l^2\right)\|x_k-x_*\|^2 + 3\alpha_k^2\|F(x_k)-F(x_*)\|^2 \\
&-2\alpha_k\langle F(x_k)-F(x_*), x_k-x_*\rangle \qquad (4.16)
\end{aligned}
$$

Setting $u = x_k$ and $v = x_*$ into (4.2) leads to the following inequality

$$\langle F(x_k) - F(x_*), x_k - x_* \rangle \geq \frac{1}{L+\mu}\|F(x_k) - F(x_*)\|^2 + \frac{L\mu}{L+\mu}\|x_k - x_*\|^2$$

and as a result of (4.16), we deduce

$$\begin{aligned} & (1-h_k)\left\|\frac{x_{k+1}-x_k}{h_k}\right\|^2 + \frac{1}{h_k}\|x_{k+1}-x_*\|^2 \\ \leq & \left(\frac{1}{h_k} + 3l^2 - 2\alpha_k\frac{L\mu}{L+\mu}\right)\|x_k - x_*\|^2 \\ & +\alpha_k\left(3\alpha_k - \frac{2}{L+\mu}\right)\|F(x_k) - F(x_*)\|^2 \end{aligned} \tag{4.17}$$

Let us first assume that $\alpha_k \leq \frac{2}{3(L+\mu)}$. Since F is strongly monotone and

$$\|F(x_k) - F(x_*)\| \geq \mu\|x_k - x_*\|,$$

we get

$$\begin{aligned} & (1-h_k)\left\|\frac{x_{k+1}-x_k}{h_k}\right\|^2 + \frac{1}{h_k}\|x_{k+1}-x_*\|^2 \\ \leq & \left(\frac{1}{h_k} + 3l^2 + 3\alpha_k^2\mu^2 - 2\alpha_k\mu\right)\|x_k - x_*\|^2. \end{aligned}$$

Having in mind $0 < h_k \leq 1$, coefficient $1 - h_k$ is positive, so ignoring the first term in the previous inequality we have

$$\|x_{k+1} - x_*\|^2 \leq \left(1 + 3l^2 h_k + 3\alpha_k^2\mu^2 h_k - 2h_k\alpha_k\mu\right)\|x_k - x_*\|^2.$$

We choose α_k such that

$$\frac{1}{3\mu}\left(1 - \sqrt{1-9l^2}\right) < \alpha_k < \frac{1}{3\mu}\left(1 + \sqrt{1-9l^2}\right), \quad k \geq 0, \tag{4.18}$$

which is equivalent to

$$A_k = 2\alpha_k\mu - 3l^2 - 3\alpha_k^2\mu^2 > 0,$$

and we also have $A_k \leq A_* = \left(\frac{2}{3} + 3l^2\right) > 0$, for $\bar{\alpha} = \frac{1}{3\mu}$. Therefore, if $l^2 < \frac{4\mu L}{9(L+\mu)^2} \leq \frac{1}{9}$, then

$$\frac{1}{3\mu}\left(1 - \sqrt{1 - 9l^2}\right) < \frac{2}{3(L+\mu)} < \frac{1}{3\mu}\left(1 + \sqrt{1 - 9l^2}\right).$$

On the other hand, if $\alpha_k \geq \frac{2}{3(L+\mu)^2}$, than again using (4.17) and Lipschitz continuity of F we get

$$\begin{aligned} &(1 - h_k)\left\|\frac{x_{k+1} - x_k}{h_k}\right\|^2 + \frac{1}{h_k}\|x_{k+1} - x_*\|^2 \\ \leq\ &\left(\frac{1}{h_k} + 3l^2 + 3\alpha_k^2 L^2 - 2\alpha_k L\right)\|x_k - x_*\|^2. \end{aligned}$$

Similarly as above, we note that $3l^2 + 3\alpha_k^2 L^2 - 2\alpha_k L < 0$, if and only if

$$\frac{1}{3L}\left(1 - \sqrt{1 - 9l^2}\right) < \alpha_k < \frac{1}{3L}\left(1 + \sqrt{1 - 9l^2}\right), \quad k \geq 0, \tag{4.19}$$

and consequently, assuming that

$$l^2 < \frac{4\mu L}{9(L+\mu)^2},$$

we can show that

$$\frac{1}{3L}\left(1 - \sqrt{1 - 9l^2}\right) < \frac{2}{3(L+\mu)} < \frac{1}{3L}\left(1 + \sqrt{1 - 9l^2}\right).$$

Summarizing, we get

$$\|x_{k+1} - x_*\|^2 \leq (1 - h_k A_k)\|x_k - x_*\|^2,$$

where for

$$\begin{aligned} t_1 &:= 1 - \sqrt{1 - 9l^2}, \\ t_2 &:= 1 + \sqrt{1 - 9l^2}, \end{aligned}$$

we have

$$A_k = \begin{cases} \alpha_k \mu(2 - 3\alpha_k \mu) - 3l^2, & \text{for } \frac{t_1}{3\mu} < \alpha_k \leq \frac{2}{3(L+\mu)} \\ \alpha_k L(2 - 3\alpha_k L) - 3l^2, & \text{for } \frac{2}{3(L+\mu)} < \alpha_k < \frac{t_2}{3L} \end{cases}.$$

It turns out that the function A_k attains the maximum value on the set

$$\frac{t_1}{3\mu} < \alpha_k < \frac{t_2}{3L}$$

at the point $\alpha_* = \frac{2}{3(L+\mu)}$ and the optimal value is

$$1 - h_k\left(\frac{4L\mu}{3(L+\mu)^2} - 3l^2\right).$$

Hence, we get

$$\|x_{k+1} - x_*\|^2 \leq \prod_{i=0}^{k}\left(1 - h_k\left(\frac{4L\mu}{3(L+\mu)^2} - 3l^2\right)\right)\|x_0 - x_*\|^2.$$

Hence, the sequence (x_k) converges to the unique solution x_* of (1.1).

$$\sum_{k=0}^{\infty} h_k = +\infty. \tag{4.20}$$

By these, the following theorem is proven:

Theorem 4. *Let H be a Hilbert space, $C_0 \subset H$ be nonempty, closed, and convex, $c : H \to H$ be $l-$Lipschitz continuous, and $C : H \to 2^H$ be a set-valued map such that $C(x) := c(x) + C_0$, for every $x \in H$. Assume that for $\mu > 0$ and $L > 0$, the map $F : H \to H$ satisfies the following condition:*

$$\|F(u)-F(v)\|^2+\mu L\|u-v\|^2 \leq (L+\mu)\langle F(u)-F(v), u-v\rangle, \quad \forall u, v \in H.$$

Parameters α_k and $0 < h_k \leq 1$, for all $k \geq 0$ satisfies (4.18), (4.19) and (4.20). Then (x_k) generated by (4.8), converges to the unique solution of (1.1).

CONCLUSION

In this paper, we have studied continuous and iterative variants of some generalizations of the gradient projection method for solving quasi-variational inequalities. We have studied the conditions under which the approximate solution obtained from the continuous and iterative methods converge to the exact solution. We have also obtained the estimates of the rate of convergence of the proposed methods. Results obtained in this paper may inspire further research in this area. The implementation and the comparison of these methods with other technique are an interesting problems for future study.

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