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ON ONE PROPERTY OF LANGRANGE MULTIPLIERS

A b s t r a c t

In this paper we consider a convex programminig problem in Hilbert space and establish one property of Lagrange multipliers. This property can be applied for construction of one numerical algorithm of minimization.

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O JEDNOM SVOJSTVU LAGRANŽOVIH MNOŽITELJA

I z v o d

U radu se razmatra zadatak konveksnog programiranja u Hilbertovom prostoru i utvrdjuju svosjstva Lagranžovih množitelja. To svojsvo može biti korišćeno za konstrukciju numeričkog algoritma minimizacije.

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0. Let H be a Hilbert space and let $f : H \to R$. Wew shall examine the following minimization problem:

$$f(x) \to \inf, x \in U = \{x \in H : -g(x) \in K\}. \tag{1}$$

where K is a closed and convex cone in normed linear space X, whose vertix is 0, and $g : H \to$X. If we define the relation $\leq$ on the space X by

$$x \leq y \text{ if and only if } y - x \in K,$$

then the set U can be written as

$$U = x \in H : g(x) \leq 0. \tag{2}$$

Note that the cone K can be described by $K = \{x \in X : 0 \leq x\}$. Its dual cone $K^* = \{\lambda \in X^* : (\forall x \in K)\langle \lambda, x\rangle \geq 0\}$ defines, in the similar way, the relation $\leq$ on the space X^*.

For the problem (1), (2) Lagrangian L is defined by

$$L(x, \lambda) = f(x) + \langle \lambda, g(x)\rangle, \; x \in H, \lambda \in K^*. \tag{3}$$

In this paper we suppoose that some of the following conditions are satisfied:

(I) *f is convex on H;*

(II) *g is convex on H;*

(III) *f is strong convex on H;*

(IV) *for every $\lambda \in K^*$, $\lambda \neq 0$, function $\varphi(x) = \langle \lambda, g(x)\rangle$ is strong convex on H;*

(V) $f, g \in C^1(H)$;

(VI) *there exists $\bar{x} \in H$ such that $-g(\bar{x}) \in int\, K$ (Slater's condition).*

Our results and numerical algorithm which will be suggested, are based on the following result (see [1]):

Theorem 1.*Assume (I),(II) and (VI). Then x_* is a solution of problem (1), (2) if and only if there exists $\lambda^* \in K^*$ such that*

$$L(x_*, \lambda^*) = \min\{L(x, \lambda^* : \; x \in H\}; \tag{4}$$

$$\langle ^*, g(x_*)\rangle = 0; \tag{5}$$

$$g(x_*) \leq 0. \tag{6}$$

If, in addition, the condition (V) be satisfied, then (4) can be replaced by

$$L'(x_*, \lambda^*) \equiv f'(x_*) + (g'(x_*))^*\lambda^* = 0. \tag{7}$$

Our aim is to solve, exactly or approximately, problem (1). In what follows we shall suggest one numerical procedure for solving it. The basical idea of this procedure is to solve the equation (7) (or to solve the minimization problem (4)) for different values of λ and to search among these values for λ whichis satisfactory for (5) and (6). The some idea was used in [3] and [4] for the construction of generalized moment method. At the begin, we shall establish some properties of the solutions of (4). Note that if the condition (III) (or (IV)) is satisfied, then problem (4) has the unique solution for every $\lambda \in K^*, \lambda \neq 0$.

In the section 1. of this paper we prove that the mapping $K^* \ni \lambda \to g(x(\lambda))$, where $x(\lambda)$ denotes the solution of (4) for given λ, is monotone and continuous. In case when the space X is real line, the corresponding theorem are proved in [3]. We describe here one numerical procedure for solving (1),(2) in this case.

In the section 2. we consider the case when the functions f and g are given by $f(x) = \|Ax - b\|^2, g(x) = \langle c, x\rangle$, where $A \in \mathcal{L}(H; H_1)$ is a continuous linear operator on a Hilbert space H to a Hilbert space H_1; $c \in H$ and $b \in H_1$. We prove that, if A is normale solvable operator then our problem has at least one solution. In this section we also show how the numerical procedure described in first section can be used in this case.

In the third section the numerical procedure is applied to a problem of minimization of terminal quadratic function on the solutions of linear differential equations system.

1. In this section we establish some properties of Lagrange multipliers.

Theorem 2. *Let $x(\lambda_1)$ and $x(\lambda_2)$ be some solutions of problem (4) for $\lambda = \lambda_1$ and $\lambda = \lambda_2$ from K^*. Then*

$$\langle g(x(\lambda_1)) - g(x(\lambda_2)), \lambda_1 - \lambda_2\rangle \geq 0. \tag{8}$$

If the functions f and g satisfay the conditions (I), (III) (or (II), (IV)), *(V) and (VI) then problem (4) has the unique solution $x(\lambda)$ for every $\lambda \in K^* \setminus \{0\}$ and the mapping $\lambda \to g(x(\lambda))$ is continuous.*

Proof. The inequalitiy (8) follows from

$$f(x(\lambda_1)) + \langle \lambda_1, g(x(\lambda_1)) \rangle \leq f(x(\lambda_2)) + \langle \lambda_1, g(x(\lambda_2)) \rangle,$$

$$f(x(\lambda_2)) + \langle \lambda_2, g(x(\lambda_2)) \rangle \leq f(x(\lambda_1)) + \langle \lambda_1 2 g(x(\lambda_1)) \rangle.$$

Namely, adding these inequalities we obtan (8).

Suppose now that the conditions (I), (III),(V) and (VI) are satisfied. Then for every $\lambda \in K^*$ mapping $x \to f(x) + \langle \lambda, g(x) \rangle$ is strong convex on H and problem (4) has the unique solution $x(\lambda)$.

Let λ belongs to K^*. Using the differential form of the conditions of convexity we obtain

$$0 \leq \langle \lambda(g'(x(\lambda))^* - \lambda(g'(x(\lambda_0))^*, x(\lambda) - x(\lambda_0) \rangle =$$

$$\langle f'(x(\lambda)) - f'(x(\lambda_0)) + (\lambda - \lambda_0) g'(x(\lambda_0)), x(\lambda) - x(\lambda_0) \rangle \leq$$

$$-\gamma \|x(\lambda) - x(\lambda_0)\|^2 + \|\lambda - \lambda_0\| \cdot \|g'(x(\lambda_0)\| \cdot \|x(\lambda) - x(\lambda_0)\|.$$

Hence,

$$\|x(\lambda) - x(\lambda_0)\| \leq \frac{1}{\gamma} \|\lambda - \lambda_0\| \cdot \|g'(x(\lambda_0).$$

It follows that $\lambda \to \lambda_0$ implies $x(\lambda) \to x(\lambda_0)$ and $g(x(\lambda)) \to g(x(\lambda_0))$. If the functions f and g satisfy the conditions (II), (IV), (V) and (VI), then the funtion $h : H \to R$ defined by $h(x) = \langle \lambda, g(x) \rangle$ is strong convex on H for every $\lambda \in K^* \setminus \{0\}$. Using again the differential form of this condition, we obtain that, for $\lambda \in K^* \setminus \{0\}$, there exists $\beta = \beta(\lambda_0) > 0$ such that

$$(\forall \lambda \in K^* \setminus \{0\})\ \beta \|x(\lambda) - x(\lambda_0)\|^2 \leq \frac{1}{\gamma} \|\lambda - \lambda_0\| \cdot \|x(\lambda) - x(\lambda_0)\| \cdot \|g'(x(\lambda_0))\|.$$

We again have that $\lambda \to \lambda_0$ implies $x(\lambda) \to x(\lambda_0)$ and $g(x(\lambda)) \to g(x(\lambda_0))$. $\square$

If the space X is equal to R then these properties and some properties proved in [2] can be efficiently used for finding satisfactory value for λ (and the corresponding solution $x(\lambda)$. Namely, then from $\lambda g(x(\lambda)) = 0$ it follows $\lambda = 0$ or $g(x(\lambda)) = 0$. If $\lambda = 0$, then the solution $x = x(0)$ of our problem (if it exists) is the solution of the problem of minimization

$f(x) \to \inf$, $x \in H$. If the function f is strong convex, this solution exists and it is unique. In case when f and g satisfy the conditions (I) and (III) the set U_* of the solutions of our problem can be empty and then $g(x(\lambda) \to +\infty$ when $\lambda \to 0$ [3]. If the set U_* is nonempty, then there exists the unique $x_* \in U_*$, such that $g(x_*) = \inf\{g(x) : x \in U_*\}$ and $\lim_{\lambda\to 0} g(x(\lambda)) = g(x_*)$ (see also [3])

Lagrange multiplier λ can be understood then as a parameter of regularization and the solutionof our problem for which $g(x)$ is minimal can be found like in the method of regularization. If the satisfactory value for λ is positive we have to solve the equation $g(x(\lambda)) = 0$. Then the mapping $\lambda \to g(x(\lambda))$ is strict monotone or $g(x(\lambda)) = const$ and $g(x(\lambda)) \to \inf\{g(x() : x \in H\} < 0$ (see also [3]). In this case, the satisfactory value for λ can be found by half-section method. It is important to note that it is not necessary to know in advance if the satisfactory value for λ is zero or positive.

2. Let H and F be Hilbert spaces and $A \in \mathcal{L}(H,F), c \in H, b \in F$ and $\beta \in R$. We shall applied the method, described in the first section, on the following minimization problem:

$$f(x) = ||Ax - b||^2,\ x \in U = \{x \in H : \langle c, x\rangle \le \beta\}. \tag{9}$$

Theorem 3. *If A is a normal solvable operator then the problem (9) has at least one solution.*

Proof. Notice that an operator A is normale solvable if the set $R(A) = \{Ax : x \in H\}$ is closed. Our proof will be based on the idea of regularization. Let (α_n) be a positive sequance such that $\alpha_n \to 0$ when $n \to \infty$. Denote the unique solution of the following problem

$$f_n(x) = ||Ax - b||^2 + \alpha_n||x||^2 \to \inf, x \in U \tag{10}$$

by x_n. It is easy to see that the sequence (Ax_n) is bounded. Prove the boundens of the sequence (x_n). It follows, by Kuhn-Tucker theorem, that there exists a real sequence (γ_n), $\gamma_n > 0$, such that

$$2A^*Ax_n + \gamma_n c + 2\alpha_n x_n = 2A^*b,\ \gamma_n(\langle c, x_n\rangle - \beta) = 0.$$

Let the set $M = \{n \in N : \gamma_n = 0\}$ be infinite. It is sufficient to consider the case $M = N$. In this case we have that, for every $n \in N$,

$$2A^*Ax_n + 2\alpha_n x_n = 2A^*b,$$

and x_n belongs to $R(A^*)$. But, if A is normal solvable then A^* is also normal solvable. Consequently, the subspace $R(A^*)$ is closed. The restriction of operator A on $R(A^*)$ has the inverse operator which is bounded. Hence, there exists a constant $m > 0$ such that

$$m\|x_n\| \leq \|Ax_n\| \leq c, \text{ for every } n \in N,$$

and the sequence (x_n) is bounded.

If the set M is finite, we can suppose that $M = \emptyset$. It follows then that for every $n \in N$

$$\gamma_n = 0, \ \langle c, x \rangle = 0.$$

But, the space H can be written as

$$H = R(A^*) + Ker\, A.$$

Hence $x_n = y_n + z_n$, where $y_n \in R(A^*)$, $z \in Ker\, A$.

The boundness of the sequence (y_n) can be proved in the similar way as a boundness of the sequence (x_n) in the first part of this proof. Let $K_n = \{z \in Ker\, A : \langle c, y_n + z \rangle = \langle c, x_n \rangle = \beta\}$. Denote by u_n the element from K_n which has the minimal norm. Then we have

$$\|u_n\| \leq m_1 |\langle c, x_n - y_n \rangle| \leq const\,.$$

Hence, the sequence (u_n) is bounded and it follows that the sequence (x'_n), $x'_n = x_n + u_n$, is bounded also. Besides, $\langle c, x'_n \rangle = \beta$, $Ax'_n = Ax_n$ and x'_n is a solution of problem (10). It means that $x'_n = x_n$ and the sequence (x_n) is bounded. From the boundness of the sequence (x_n) it follows that it contains a subsequence (x_{n_k}) which weakly converges to $x_* \in U$. Using continuity and convexity of the function f, we conclude that $f(x_{n_k}) \to f(x_*)$. But, then $f_{n_k}(x_{n_k}) \to f(x_*)$ also. By these facts and by theorem of regularization [2], it follows that x_* is the solution of (9) with the smallest norme. □

Observe that our functions $f(x) = \|Ax - b\|^2$ and $g(x) = \langle c, x \rangle$ are not strong convex and the corresponding equation (7)

$$A^*Ax + \lambda c = A^*b \tag{7'}$$

need not have any solution for $\lambda = 0$. Moreover, if c does not belong to $R(A^*)$ then this equation has not any solution. For this reason we shall solve the regularized equation

$$A^*Ax + \alpha x + \lambda c = A^*b \tag{10'}$$

where α is positive but closed to zero. We shall consider this equation with the conditions

$$\lambda(\langle c, x\rangle - \beta) = 0, \lambda \geq 0, \langle c, x\rangle \leq \beta.$$

This is connected with the following minimization problem

$$||Ax - b||^2 + \alpha||x||^2 \to \inf, x \in U. \tag{11}$$

If we denote its unique solution by x_α and the solution of (9) with the smallest norm by x_*, then [2]

$$||x_\alpha - x_*|| \leq const \cdot \alpha. \tag{12}$$

Approximation $\overline{x_\alpha}$ of the solution x_α can be found by method described before. From the estimate (12) it follows that $x_\alpha \to x_*$ when $\alpha \to 0$ and we can accept $\overline{x_\alpha}$ as a approximation for x_α.

Problem of the choise of value for the parametar of regularization α was considered in [3]. It was supposed there that instead of the operator A and the vector b are known only their approximations. These results can be applied to our problem also.

3. In order to ilustrate our method we shall consider the following optimal control problem:

$$f(x) = |y(T, x) - z|^2 \to \inf,\ x \in U = \{x \in L_2^r[0, T] : \langle c, x\rangle \leq \beta\}. \tag{13}$$

Here $y(\cdot, x)$ is the solution of Cauchy problem

$$y' = P(t)y(t) + Q(t)x(t) + g(t), 0 < t < T, \tag{14}$$

$$y(0) = y_0. \tag{15}$$

We suppose that $P = P(\cdot) = (p_{ij}(\cdot))_{n\times n}$, $Q = Q(\cdot) = (q_{ij}(\cdot))_{n\times r}$, $g = g(\cdot) = (g_i(\cdot))_n$ are given matrices whose elements belong to $L_\infty[0, T]$. We also suppose that the finite moment $T > 0$, the begining and desired states $y_0, z \in R^n$ are given. Then, for every $x \in L_2^r[0, T]$, there exists the unique absolutely continuous function $y = y(\cdot)$ defined on the whole $[0, T]$ so that $y(0) = y_0$ and (14) holds almost everywhere in $(0, T)$.

The problem (13)-(15) can be writen as

$$f(x) = ||Ax - b||^2 \to \inf,\ x \in U = \{x \in L_2^r[0, T] : \langle c, x\rangle \leq \beta\}$$

where $A : H = L_2^r[0,T] \to R^n$ is given by

$$Ax = y(T, x, g = 0, y_0 = 0)$$

and

$$b = z - y(T, x = 0, g, y_0).$$

Hence, problem (13)-(15) is a special case of the problem (9). Operator A is normal solvable and this problem has at least one solution. If we denote the unique solution of equation (10') (for fixed $\alpha > 0$ and $\lambda \geq 0$) by $x_\alpha(\lambda)$, then $\alpha x_\alpha(\lambda) + \lambda c$ belongs to $R(A^*)$. But, the subspace $R(A^*) = \{A^*x : x \in R^n\}$ is finitedimensional and it is generated by vectors $h_i = A^*e_i$, where $\{e_i : i = 1, \ldots, n\}$ is a base of the space R^n. Operator $A^* : R^n \to L_2^r[0,T]$ is defined by

$$(A^*q)(\cdot) = Q^T(\cdot)\varphi(\cdot),\ \varphi'(t) = -P^T(t)\varphi(t),\ 0 < t < T,\ \varphi(T) = q \in R^n.$$

Hence, $x_\alpha(\lambda)$ can be written as

$$x_\alpha(\lambda) = -\frac{\lambda}{\alpha}\sum_{i=1}^{n} \xi_i h_i. \tag{16}$$

From (10') and (16) we obtain that the real numbers ξ_i must satisfy the following equation

$$\sum_{i=1}^{n} \xi_i (A^*A + \alpha I)h_i = A^*b - \frac{\lambda}{\alpha} A^*Ac.$$

Multiplying this equation by $h_j, j = 1, \ldots, n$, we have

$$\sum_{i=1}^{n} \xi_i(\langle Ah_i, Ah_j\rangle + \alpha\langle h_i, h_j\rangle) = \langle f, Ah_j\rangle - \frac{\lambda}{\alpha}\langle Ac, Ah\rangle,\ j = 1, \ldots, n.$$

We have just obtained the system of linear equations and we have to solve it for different value of λ and search λ which is satisfactory for (11). It means that we have to solve many different linear systems. But, all this sistems (for fixed α) have the same matrice and only the right side is changable. Morever, the changes in the right side are very simple. In our numerical experiment we was computing vectors h_i and scalar products in L_2^r using the simpliest Euler's method for solving corresponding systems of differential equation and Simpson's rule for computation corresponding integrals.

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