ЦРНОГОРСКА АКАДЕМИЈА НАУКА И УМЈЕТНОСТИ
ГЛАСНИК ОДЈЕЉЕЊА ПРИРОДНИХ НАУКА, 25, 2022.

ЧЕРНОГОРСКАЯ АКАДЕМИЯ НАУК И ИСКУССТВ
ГЛАСНИК ОТДЕЛЕНИЯ ЕСТЕСТВЕННЫХ НАУК, 25, 2022

THE MONTENEGRIN ACADEMY OF SCIENCES AND ARTS
PROCEEDINGS OF THE SECTION OF NATURAL SCIENCES, 25, 2022

UDK 512.541.1

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TORUS ACTIONS IN TOPOLOGY

Abstract

The study of group actions is now widely established in all areas of mathematics mainly from two aspects. The study of the orbit spaces of group actions from the point of view of topology, algebra, geometry, combinatorics shows up to be important in series of problems in mathematics and mathematical physics. On the other hand, the presence of a group action on a topological, algebraic, geometrical or combinatorics object and study of features of that action leads in many cases to essential results about an initial object. In this paper we review some of results about compact torus actions on smooth manifolds. A special attention is devoted to the results on the canonical action of the compact torus on compact homogeneous spaces of positive Euler characteristic. It is mainly related to the theory of unitary cobordisms of these spaces as well as to the construction of the combinatorial-smooth models for their orbit spaces.

2010 Mathematics Subject Classification. 57R19, 57R77, 57R91, 57S12
Key words and phrases: manifolds, torus actions, complex cobordisms, orbit spaces

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1. Introduction

The idea of a group G acting on a set of elements X dates back to Galois and his breakthrough in development of abstract algebra in the 1830s. The well known Erlangen program of Klein established this idea forty years later in the case when X denotes the set of points and G a group of isometries, and proved to be fundamental for development of geometry. In the sequent development of almost all fields of mathematics the idea of a group action has shown to bring as well the breakthrough results and open new directions for development.

The study of torus actions on topological spaces is one of the most current issues in equivariant topology. The specific problems related to torus actions arise in many fields of mathematics and mathematical physics and many publications have been devoted to various aspects of this topic. Topological approach to this issue is presented in famous monograph of G. Bredone [9], in monograph of Audin [3] it is presented an approach to this subject from the point of view of symplectic geometry, while an algebro-geometrical point of view is given through several fundamental research papers and monographs such as works of Danilov [23], Oda [45], Fulton [27], Ewald [26]. In recent monograph Toric topology by Buchstaber and Panov [12] it is presented a comprehensive overview of the previous as well as modern results on this subject.

The orbit spaces of various torus actions may have very rich combinatorial structure. In many cases the study of combinatorics of an orbit space is an efficient way in understanding an initial torus action and the vise versa holds as well. The study of equivariant topology of torus actions very often was leading to topological proofs of some highly non-trivial results in algebraic topology and combinatorics. For example, in the paper of Davis and Januskiewitz [24] as well as in the series of papers by Buchstaber and Panov, it has been developed the theory of quasitoric manifolds M^{2n} with an effective action of the torus T^n. In this case the orbit space M^{2n}/T^n is homeomorphic to a simple convex polytope P and the cohomology ring for M^{2n} is determined by the combinatorics of the polytope P. In addition, the initial manifold M^{2n} can be reconstructed by this polytope as so called characteristic function which assigns to any point from M^{2n} its stabilizer for this action.

On the other hand, in the case of a group action with isolated fixed points various algebro-topological invariants of an initial manifold can be expressed in terms of the local data for the action at the fixed points. For example, this can be done for the complex cobordism classes of manifolds with such a torus action due to the achievement of Novikov [44], as well as for the general characteristic classes and in that context various cobordism classes on compact homogeneous spaces using, at the first place, the results of Borel-Hirzebruch [7], [8].

In this paper, we review some of the achievements which demonstrate the power of the techniques of toric topology in studying some fundamental questions of algebraic topology such as description of complex cobordism classes and Hirzebruch genera on manifolds with torus action, in particular homogeneous space. We also outline the theory of $(2n, k)$-manifolds recently developed by Buchstaber-Terzić in [20], in which the tools for an effective description of equivariant structure of such manifolds as well as the structure of their orbit spaces are proposed. The class of these manifolds includes all well known classes of manifold with nicely behaved torus actions and their generalizations.

2. Unitary cobordisms

Cobordisms are one of the central objects of study in geometric topology and algebraic topology. In geometric topology cobordisms are closely connected to Morse theory and surgery theory, while in algebraic topology, cobordisms theories are fundamental cohomology theories and categories of cobordisms are the basic objects in topological quantum filed theory.

Two stable complex smooth, closed manifolds M_1^n and M_2^n are said to be (co)bordant if there is a manifold with boundary W^{n+1} such that $\partial W^{n+1} = M_1^n \sqcup M_2^n$. Complex bordism classes form the complex bordism ring $\Omega_*^U = \Omega_*^U(pt)$ with respect to the disjoint union and product. It is the result of Milnor- Novikov [42], [43] that

$$\Omega_*^U \cong \mathbb{Z}[a_1, a_2, \ldots], \quad \deg a_i = 2i.$$

Toric methods have wide applications in complex cobordism theory. The well known question in complex cobordism theory is to describe the multiplicative generators for the complex cobordism ring Ω_*^U and the manifolds with nice properties which represent corresponding bordism classes. From the point of view of toric topology such nice manifolds are manifolds with well behaving torus actions which preserve a stable complex structure. It is known that the Milnor surfaces H_{ij}, $0 \leq i \leq j$ give the mostly known generators for the complex cobordism ring Ω_*^U, [42], but there is no action on Milnor surfaces making them into quasitoric manifolds, [11]. In this context we also point to Milnor's theorem, which states that any bordism class in Ω_n^U with $n > 0$ contains a nonsingular algebraic variety not necessarily being connected, [42]. Milnor surfaces are nonsingular algebraic varieties, but it is still left to find nonsingular algebraic representative for $-[H_{ij}]$, see [49]. Related to this we mention the following question raised by Hirzebruch in the late 1950's: describe the set of bordism classes in Ω_*^U which can be represented by connected nonsingular algebraic varieties. This question is still open even in complex dimension 2.

The mostly well behaved torus actions are on toric and quasitoric manifolds. A family of toric manifolds B_{ij}, $0 \leq i \leq j$ which multiplicatively generate complex cobordism ring Ω_*^U is constructed by Buchstaber - Ray in [14]. It implies that any complex cobordism class contains a disjoint union of toric manifolds. Related to the Hirzebruch's question, connected representatives in cobordism classes can not be found among toric manifolds due to certain topological obstructions [12]. In the class of quasitoric manifolds this is possible to do due to the result of Buchstaber-Panov-Ray [13], which states that for $n > 2$ every complex cobordism class contains a quasitoric manifold, necessarily connected, whose stably complex structure is compatible with the action of the torus. In other words any stable complex manifold is cobordant to a manifold with nicely behaving torus action.

We recall that the generators in Ω_*^U can be detected by the characteristic numbers. For a complex k-vector bundle over a manifold M it is formally defined the total Chern class by

$$c(\xi) = 1 + c_1(\xi) + \ldots + c_k(\xi),$$

where $c_i(\xi) = \sigma_i(x_1, \ldots, x_k)$ is the i-the elementary symmetric function in formal variables $x_1, \ldots, x_k$. These variables have a geometric meaning in the following sense: if $\xi = \xi_1 \oplus \cdots \oplus \xi_k$ for line bundles $\xi_1, \ldots, \xi_k$, then $x_j = c_1(\xi_j)$, $1 \leq j \leq k$ is the first Chern class of ξ_j. Let us consider the polynomial $P_n(x_1, \ldots, x_k) = x_1^n + \ldots + x_k^n$ and write it in terms of elementary symmetric functions, that is, $P_n(x_1, \ldots, x_k) = \sigma_n(\sigma_1(x_1, \ldots, x_k), \ldots, \sigma_k(x_1, \ldots, x_k))$. Substituting the Chern classes $c_i(\xi)$ for the elementary symmetric functions σ_i we obtain some characteristic class

$$s_n(\xi) = P_n(c_1(\xi), \ldots, c_k(\xi)).$$

It turns out that this characteristic class plays an important role in detecting the polynomial generators of the complex cobordism ring due to the following result [42], [43]. A stable complex structure c_τ on a manifold M^{2n} is a real isomorphism $c_\tau : \tau(M^{2n}) \oplus \mathbb{R}^{2l} \to \xi$, where ξ is a complex vector bundle over M^{2n}. The structure c_τ said to be an almost complex structure if $l = 0$. Let

$$s_n[M^{2n}] = s_n(\xi)\langle M^{2n}\rangle \in \mathbb{Z},$$

where ξ is as above and $\langle M^{2n}\rangle \in H_{2n}(M^{2n})$ is the fundamental class for M^{2n}.

Theorem 1. *A bordism class $[M] \in \Omega_{2n}^U$ represents a polynomial generator in the ring Ω_*^U if and only if*

$$s_n[M] = \begin{cases} \pm 1 & \textit{if} \quad n \neq p^k - 1 \quad \textit{for any prime } p \\ \pm p & \textit{if} \quad n = p^k - 1 \quad \textit{for some prime } p \end{cases}$$

The number $s_n[M^{2n}]$ is sometimes called the magic number of a manifold M^{2n}.

3. Universal toric genus

The paper [44] of S. P. Novikov opened a new stage in the development of the cobordism theory by proposing a method for the description of the fixed points for actions of groups on manifolds, based on the formal group law for geometric cobordisms. It quickly stimulated active research work on cobordisms of manifolds with group actions. One branch of this direction is the theory of equivariant genera for stable complex manifolds equipped with compatible actions of the torus T^k, which was introduced by Buchstaber-Ray in [15] and developed in detail in [13]. This theory is based on the notion of universal toric genus, which represents an equivariant analogue of an universal Hirzebruch genus. A universal toric genus is a genus defined on a class of stable complex T^k-manifolds and taking values in the complex cobordism ring $U^*(BT^k)$ of the classifying space of the torus T^k. We recall shortly its definition for a manifold M^{2n} endowed with a tangentially stable complex structure c_τ and a smooth action θ of the torus T^k. A stable complex structure is said to be compatible with an action θ if the transformation given by

$$r(t) : \xi \overset{c_\tau^{-1}}{\to} \tau(M^{2n}) \oplus \mathbb{R}^{2l} \overset{d\theta(t)\oplus I}{\to} \tau(M^{2n}) \oplus \mathbb{R}^{2l} \overset{c_\tau}{\to} \xi$$

is a complex transformation for any $t \in T^k$. The Borel construction gives the fibration

$$M^{2n} \to ET^k \times_{T^k} M^{2n} \to\overset{p}{\to} BT^k,$$

whose tangent bundle along the fibers is naturally endowed with the stable complex structure, where ET^k and BT^k stand for the universal, respectively classifying space of the torus T^k. Consider now the Gysin homomorphism

$$p_! : U^*(ET^k \times_{T^k} M^{2n}) \to U^*(BT^k),$$

where $U^*(X)$ denotes the theory of unitary cobordisms of the space X, that is the generalized cohomology theory of X with values in Ω_*^U. *The universal toric genus* of the triple (M^{2n}, c_τ, θ) is defined by

$$\Phi(M^{2n}, c_\tau, \theta) = p_!(1).$$

Recall that $U^*(BT^k) = \Omega_*^U[[u_1, \dots, u_k]]$ is the algebra of formal power series over Ω_*^U. The following expression for the universal toric genus is due to Buchstaber-Ray [15]:

$$\Phi(M^{2n}, c_\tau, \theta) = [M^{2n}] + \sum_{|\omega|>0} [G_\omega(M^{2n}]\mathbf{u}^\omega, \tag{1}$$

where $\omega = (i_1, \dots, i_k)$ and $\mathbf{u}^\omega = u_1^{i_1} \cdots u_k^{i_k}$. The expression $[M^{2n}]$ denotes the complex cobordism class of (M^{2n}, c_τ), while G_ω is certain stable complex manifold obtained as the total space of the fibration with the fiber M^{2n}.

In the case when all fixed points for T^k-action on M^{2n} are isolated, it is proved in [13], [15] that the universal toric genus for (M^{2n}, c_τ, θ) can be localized, that is expressed in terms of the local data at the fixed points for the action θ related to c_τ. Let P denotes the set of fixed points and $\{\Lambda_j(p), 1 \leq j \leq n\}$, $\Lambda_j(p) \in \mathbb{Z}^k$ be the set of weight vectors of the representation of T^k at the tangent space T_pM^{2n} related to the stable complex structure c_τ for $p \in P$. Further, let $[\Lambda_j(p)]$ is defined by the power series of the formal group low in complex cobordisms, see [16]. The localization theorem states:

Theorem 2. *If all fixed points for the action θ are isolated then*

$$\Phi(M^{2n}, c_\tau, \theta) = \sum_{p\in P} sign(p) \prod_{j=1}^{n} \frac{1}{[\Lambda_j(p)](\mathbf{u})},$$

where $sign(p)$ denotes the sign of a fixed point p related to the stable complex structure c_τ.

To obtain a formula from which one can derive more explicit results, it can be employed the notion of Chern-Dold character following [10]. The Chern-Dold character of a topological space X in the theory of unitary cobordisms U^* is a ring homomorphism

$$ch_U : U^*(X) \to H^*(X, \Omega_U^* \otimes \mathbb{Q}).$$

Since the Chern-Dold character splits into the composition $ch_U : U^*(X) \to H^*(X, \Omega_U^*(\mathbb{Z})) \to H^*(X, \Omega_U^* \otimes \mathbb{Q})$, it can be considered as a ring homomorphisms $ch_U : U^*(X) \to H^*(X, \Omega_U^*(\mathbb{Z}))$, where $\Omega_U^*(\mathbb{Z})$ is a subring of $\Omega_U^* \otimes \mathbb{Q}$ given by $\Omega_U^*(\mathbb{Z}) = \mathbb{Z}[b_1, \ldots, b_n, \ldots]$ for $b_n = \frac{1}{n+1}[\mathbb{C}P^n]$. Being multiplicative transformation of cohomology theories, it is then given by the series

$$ch_U u = h(x) = \frac{x}{f(x)}, \quad f(x) = 1 + \sum_{i=1}^{\infty} a_i x^i \ \text{ for } \ a_i \in \Omega_U^{2i}(\mathbb{Z})$$

Here $u = c_1^U(\eta) \in U^2(\mathbb{C}P^\infty)$ and $x = c_1^H(\eta) \in H^2(\mathbb{C}P^\infty, \mathbb{Z})$ denote the first cobordism, respectively cohomology Chern class of the universal complex line bundle $\eta \to \mathbb{C}P^\infty$.

From the construction of the Chern-Dold character it follows

$$ch_U[M^{2n}] = [M^{2n}] = \sum_{\|\omega\|=n} s_\omega([M^{2n}])a^\omega,$$

where $\omega = (i_1, \ldots, i_n)$, $\|\omega\| = \sum_{l=1}^{n} l i_l$ and $a^\omega = a_1^{i_1} \cdots a_n^{i_n}$. The numbers $s_\omega(\tau(M^{2n})$ are the cohomology characteristic numbers for M^{2n} and they correspond to the tangent characteristic classes for M^{2n}.

Applying the Chern-Dold character to the universal toric genus in the case when localization formula holds, one obtains

$$(2)\qquad [M^{2n}] + \sum_{|\omega|>0} [G_\omega(M^{2n}](ch_U\mathbf{u})^\omega = \sum_{p\in P} \mathrm{sign}(p) \prod_{j=1}^{n} \frac{f(\langle \Lambda_j(p), \mathbf{x}\rangle)}{\langle \Lambda_j(p), \mathbf{x}\rangle},$$

where $ch_U\mathbf{u} = (ch_U u_1, \ldots, ch_U u_k)$, $ch_U u_i = \frac{x_i}{f(x_i)}$, $\omega = (\omega_1, \ldots, \omega_k)$, $|\omega| = \omega_1 + \ldots + \omega_k$ and $\mathbf{x} = (x_1, \ldots, x_k)$ and $\langle \Lambda_j(p), \mathbf{x}\rangle = \sum_{l=1}^{k} \Lambda_j^l(p) x_l$ for $\Lambda_j(p) = (\Lambda_1^1(p), \ldots, \Lambda_j^k(p))$.

This equality leads to the expression for the complex cobordism class of $[M^{2n}]$:

Proposition 1. *The coefficient for t^n in the series in t^n*

$$\sum_{p\in P} sign(p) \prod_{j=1}^{n} \frac{f(t\langle \Lambda_j(p), \mathbf{x}\rangle)}{\langle \Lambda_j(p), \mathbf{x}\rangle}$$

represents the complex cobordism class $[M^{2n}]$, while the coefficient in t^l for $0 \leq l \leq n-1$ is equal to zero.

Now set $t_j(p) = \langle \Lambda_j(p), \mathbf{x}\rangle$, $1 \leq j \leq n$ and set

$$\prod_{j=1}^{n} f(t_j) = 1 + \sum_{\omega} f_\omega(t_1, \ldots, t_n) a^\omega.$$

Together with (2) we obtain an explicit formula for cohomology characteristic numbers:

Theorem 3. *For any $\omega = (i_1, \ldots, i_n)$ such that $\|\omega\| = \sum_{l=1}^{n} l \cdot i_l = n$ it holds*

$$s_\omega[M^{2n}] = \sum_{p\in P} sign(p) \cdot \frac{f_\omega(t_1(p), \ldots, t_n(p))}{t_1(p)\cdots t_n(p)}.$$

In particular, this implies that the characteristic number $s_n[M^{2n}]$, which as we already said, plays significant role in determining polynomial generators of the ring Ω_*^U can be expressed by:

Corollary 1.

$$s_{(0,\ldots,0,1)}[M^{2n}] = s_n[M^{2n}] = \sum_{p\in P} sign(p) \frac{\sum_{j=1}^{n} t_j^n(p)}{t_1(p)\cdots t_n(p)}.$$

3.1. **Application to homogeneous spaces of positive Euler characteristic.** We consider homogeneous spaces G/H of positive Euler characteristic, meaning that G and H have the common maximal torus T^k, and we assume G/H to be endowed with an G - invariant almost complex structure J. For the background on necessary Lie theory of compact homogeneous spaces we refer to [7], [8]. The following holds:

- The fixed points for the canonical T^k-action on G/H are given by the set $W_G \cdot H$, where W_G is the Weyl group for G. Consequently, the number of fixed points is equal to the Euler characteristic $\chi(G/H) = \frac{|G|}{|H|}$.
- If $\alpha_1, \ldots, \alpha_n$ are the complementary roots for G related to H, then any invariant almost complex structure on G/H can be uniquely described by the root system $\Lambda_1 = \varepsilon_1\alpha_1, \ldots, \Lambda_n = \varepsilon_n\alpha_n$, where $\varepsilon_i = \pm 1$ depending on certain orientation issue. The roots $\{\Lambda_k\}$ are called the roots of an almost complex structure J.
- The weights of the canonical T^k-action on G/H at the fixed point $p = e$ are given by the roots of the invariant almost complex structure J, that is by $\Lambda_1, \ldots, \Lambda_n$.
- The weights at a fixed point $\mathrm{w}\cdot H$ for $\mathrm{w} \in W_G$ are given by the action of w on the complementary roots for G related to H, that is by $\mathrm{w}(\Lambda_1), \ldots, \mathrm{w}(\Lambda_n)$.
- The sing of any fixed point for the canonical T^k-action on G/H related to an invariant almost complex structure is $+1$.

Consequently, for the cobordism class and cohomology characteristic numbers the following holds:

Theorem 4. *The cobordism class for $(G/H, J)$ is given as the coefficient for t^n in the series in t*

$$\sum_{\mathrm{w}\in W_G/W_H} \prod_{j=1}^{n} \frac{f(t\langle \mathrm{w}(\Lambda_j), \mathbf{x}\rangle)}{\langle \mathrm{w}(\Lambda_j), \mathbf{x}\rangle}$$

Theorem 5. *The cohomology characteristic numbers for $(G/H, J)$ are given by*

$$s_\omega(\tau(M^{2n})) = \sum_{\mathrm{w}\in W_G/W_H} \mathrm{w}\Big(\frac{f_\omega(t_1, \ldots, t_n)}{t_1 \cdots t_n}\Big),$$

where $t_j = \langle \Lambda_j, \mathbf{x}\rangle$, $1 \leq j \leq n$. Consequently,

Corollary 2.

$$s_n(G/H, J) = \sum_{\mathrm{w}\in W_G/W_H} \mathrm{w}\Big(\frac{\sum_{j=1}^{n} t_j^n}{t_1 \cdots t_n}\Big).$$

Remark 1. Homogeneous spaces G/H for which $\mathrm{rk}\, H < \mathrm{rk}\, G$ or what is equivalent to saying that Euler characteristic for G/H is zero, are not interesting from the point of view of equivariant cobordism theory, as they are equivariantly cobordant to zero [17].

Employing some additional techniques, like divided difference operators, in [16] and [17] are computed the characteristic numbers and cobordism classes of some important classes of homogeneous spaces.

3.1.1. *Complex flag manifolds.* A flag in $\mathbb{C}^n$ is a sequence of increasing subspaces $0 = V_0 \subset V_{n_1} \subset V_{n_2} \subset \ldots \subset V_{n_k} = V_n = \mathbb{C}^n$, where $\dim V_{n_i} = n_i$. For a fixed sequence of integers $n_1, \ldots, n_k = n$ the set of all such flags is a manifold $F_{n_1,\ldots,n_k}$. The flags corresponding to the sequence $1, 2, \ldots, n$ are called complete flags and $F_{1,2\ldots,n}$ is called the full flag manifold. The unitary group $U(n)$ acts transitively on $F_{1,2,\ldots,n}$ with the stabilizer T^n, so the full flag manifold can be represented as a homogeneous space, that is $F_{1,2\ldots,n} = U(n)/T^n$. It follows from [8] that $U(n)/T^n$ admits 2^m, $m = \frac{n(n-1)}{2}$ invariant almost complex structures and only two of them, conjugate to each other, are integrable, that is they are invariant complex structure. Applying the above theory, it is derived in [16] that the cobordism classes of $U(n)/T^n$ for the invariant complex structure are given by

$$[U(n)/T^n] = \sum_{\sigma\in S_n} \mathrm{sign}(\sigma) P_{\sigma(\delta)}(a_1, \ldots, a_n, \ldots),$$

where S_n is the symmetric group, while the polynomials $P_{\sigma(\delta)}(a_1, \ldots, a_n, \ldots)$, $\delta = (n-1, n-2, \ldots, 1, 0)$ are defined by

$$\prod_{1\leq i<j\leq n} f(t(x_i - x_j)) = 1 + \sum_{|\xi|>0} P_\xi(a_1, \ldots, a_n, \ldots)t^{|\xi|}\mathbf{x}^\xi,$$

for $\xi = (j_1, \ldots, j_n) \in \mathbb{Z}^n_{\geq 0}$ and $|\xi| = \sum_{q=1}^{n} j_q$. In addition, the top characteristic number is:

$$s_m(U(n)/T^n = 0, \text{ for } n > 3, \ m = \frac{n(n-1)}{2}, \ s_3(U(3)/T^3) = -6.$$

3.1.2. *Complex Grassmann manifolds.* A complex Grassmann manifold $G_{q+l,l}$ consists of all l-dimensional complex subspaces in $\mathbb{C}^{q+l}$, where the integers $q, l \geq 1$. In particular for $l = 1$ we have that $G_{q+1,1} = \mathbb{C}P^q$ is the complex projective spaces. The unitary group $U(q + l)$ acts transitively on $G_{q+l,l}$ with stabilizer $U(q) \times U(l)$, so $G_{k+l,l}$ can be represented as a homogeneous space, that is $G_{q+l,l} = U(q + l)/U(q) \times U(l)$. Being irreducible homogeneous space it follows from Borel-Hirzebruch that Grassmannian $G_{q+l,l}$ admits, up to equivalence, one invariant complex structure. It is proved in [16] that

$$[G_{q+l,l}] = \frac{1}{q!l!}\sum_{\sigma\in S_{q+l}} \text{sign}(\sigma)Q_{(q+l,l)\sigma(\delta)}(a_1,\ldots,a_{ql}),$$

for $\delta = (q+l-1, q+l-2,\ldots,0)$, and the polynomials $Q_{(q+l,l)\sigma(\delta)}(a_1,\ldots,a_{ql})$ are defined by

$$\Delta_q\Delta_{q+1,q+l}\prod_{1\leq i\leq q//q+1\leq j\leq q+l} f(t(x_i-x_j)) = \sum_{|\xi|\geq\frac{(q+l)^2-(q+l)}{2}} Q_{(q+l,l)\xi}(a_1,\ldots,a_n,\ldots)t^{|\xi|-\frac{(q+l)^2-(q+l)}{2}}x^{\xi},$$

for $\Delta_{p,q} = \prod\limits_{p\leq i<j\leq q}(x_i-x_j)$, $\Delta_n = \Delta_{n,1}$ and ξ is as in the previous example.

The top characteristic number is given by

$$s_{lq}(G_{q+l,l}) = \sum_{\sigma\in S_{q+l}/(S_q\times S_l)}\sigma\Big(\frac{\sum(x_i-x_j)^{lq}}{\prod(x_i-x_j)}\Big).$$

Moreover, it is proved in [17] that

$$s_{lq}(G_{q+l,l}) = 0 \ \text{ for } \ q,l\geq 3.$$

For $q = l = 2$ we obatin $s_4(G_{4,2}) = -20$, while for $l = 2, q = 3$ we obtain $s_3(G_{5,2}) = 70$.

3.1.3. *Generalized Grassmann manifolds.* The cobordism classes and characteristic numbers of generalized Grassmann manifolds $G_{q_1+q_2+\ldots q_k,q_1,\ldots,q_{k-1}}$ are also calculated in [17]. These manifolds can be represented as homogeneous spaces, that is $G_{q_1+q_2+\ldots q_k,q_1,\ldots,q_{k-1}} = U(q_1+\ldots+q_k)/U(q_1)\times\ldots\times U(q_k)$. A generalized Grassmann manifold has $2^{\frac{k(k-1)}{2}}$ invariant almost complex structures. Let J be an invariant almost complex structure defined by the roots $\varepsilon_{ij}(x_i-x_j)$, where $q_1+\ldots+q_{l-1}+1\leq i\leq q_1+\ldots+q_l$ and $q_1+\ldots+q_l+1\leq j\leq q_1+\ldots+q_k$ for $1\leq l\leq k-1$, and $\varepsilon_{ij} = \pm 1$. The cobordism class of $G_{q_1+\ldots+q_k,q_1,\ldots,q_{k-1}}$ is given as the coefficient for t^m, $m = \sum\limits_{1\leq i<j\leq k} q_iq_j$ in the series in t:

$$\frac{\lambda}{q_1!\cdots q_k!}L\big(\Delta_{q_1}\times\Delta_{q_1+1,q_1+q_2}\times\ldots\times\Delta_{q_1+\ldots+q_{k-1}+1,q_1+\ldots+q_k}\prod f(t\varepsilon_{ij}(x_i-x_j))\big),$$

where $\lambda = \prod\varepsilon_{ij}$, $\Delta_{p,q} = \prod\limits_{p\leq i<j\leq q}(x_i-x_j)$ and L denotes divided difference operator. The operator L is defined by $L\mathbf{x}^{\xi} = \frac{1}{\Delta_n}\sum\limits_{\sigma\in S_n}\text{sign}(\sigma)\sigma(\mathbf{x}^{\xi})$, for $\xi = (j_1,\ldots,j_n)$ and $\mathbf{x}^{\xi} = x_1^{j_1}\cdots x_n^{j_n}$.

It implies that

$$s_m(G_{q_1+\ldots+q_k,q_1,\ldots,q_{k-1}}) = 0 \text{ for } k \geq 3, \text{ where } m = \sum_{1\leq i<j\leq k} q_i q_j.$$

4. Hirzebruch genera

The Hirzebruch genera have been introduced in algebraic geometry and became widely known due to the famous Atiyah-Singer theorem on index of differential operators of manifolds. They had evolved out of the classical concept of genus of a surface which dates back to Riemann [47]. A genus is an invariant for certain classes of manifolds expressed in terms of characteristic classes of the tangent bundle, which might be equipped with a stable complex structure, with values in a ring R and it satisfies two properties. The first one is additivity, which means that a genus of the disjoint union of two manifolds is equal to the sum of same genera and the second is multiplicativity, which means that a genus of the product of two manifolds is the product of the same genera. A beautiful explanation of the concept of genus is given by Hirzebruch and Kreck in [34]. The Hirzebruch genera play fundamental role in complex cobordism theory and its applications.

Le us recall the notion of a Hirzebruch genus. Assume we are given a power series $f(x) = x + f_1x^2 + f_2x^3 + \ldots \in R \otimes \mathbb{Q}[[x]]$, where R is a commutative ring with unit. It is common as well to assume that R is torsion-free due to the beautiful Hirzebruch correspondence that follows. The formal series

$$\prod_{i=1}^{n} \frac{u_i}{f(u_i)}$$

is a symmetric function in variables $u_1, \ldots, u_n$, so it can be represented in the form $\mathcal{L}_f(\sigma_1, \ldots, \sigma_n)$ for the elementary symmetric functions $\sigma_1, \ldots, \sigma_n$ in variables $u_1, \ldots, u_n$. The Hirzebruch genus $\mathcal{L}_f(M^{2n})$ of a stable complex manifold M^{2n} is defined to be the value of $\mathcal{L}_f(c_1, \ldots, c_n)$ on the fundamental class $\langle M^{2n}\rangle$, where c_i are the Chern classes of the corresponding complex vector bundle ξ over M^{2n}. A Hirzebruch genus is a multiplicative invariant of a stable complex manifold, that is, it defines the ring homomorphism $\mathcal{L}_f : \Omega_*^U \to R \otimes \mathbb{Q}$. The vice verse is also true: for any ring homomorphism $\phi : \Omega_*^U \to R \otimes \mathbb{Q}$, there exists a series $f \in R \otimes \mathbb{Q}[[u]]$, such that $f(0) = 0$, $f'(0) = 1$ and $\phi = \mathcal{L}_f$. The formal power series f is called the logarithm of ϕ. Among Hirzebruch genera in the focus of study, due to their importance in different areas of mathematics and mathematical physics, are classical genera such as the Todd genus, the signature and the arithmetic genus, as well as the modern genera such as an elliptic genus, the Krichever genus and the general Krichever genus.

For example, may be the best known genus is the signature of a $2n$-dimensional manifold, which is defined to be the signature of the intersection form of its cohomology algebra for even n, while it is equal to zero for odd n. It is known that the signature is an essentially important topological invariant of a four-manifold, for example, together with the rank it classifies the rational homotopy type [51] of a four-manifold. The signature can be defined as the L-genus, whose logarithm is given by the power series

$$f(u) = u + \frac{u^3}{3} + \frac{u^5}{5} + \ldots = tanh^{-1}(u).$$

We further assume, without loss of generality that R is an $\mathbb{Q}$-algebra. Any Hirzebruch genus $\mathcal{L}_f : \Omega_*^U \to R$ has T^k-equivariant extension $\mathcal{L}_f^{T^k} : \Omega_*^{U:T^k} \to R[[u_1, \ldots, u_k]]$ defined by

$$\mathcal{L}_f^{T^k} = \mathcal{L}_f \circ \Phi,$$

where $\Omega_*^{U:T^k}$ denotes the cobordism classes of T^k-equivariant stable complex manifolds and Φ is the universal toric genus. In this way the Hirzebruch genera give the class of toric genera. It follows from (1) that

$$\mathcal{L}_f^{T^k}(M, c_\tau) = \mathcal{L}_f(M) + \sum_{|\omega|>0} \mathcal{L}_f(G_\omega)\mathbf{u}^\omega.$$

We present some of the results from [16], [17] on some Hirzebruch genera on compact homogeneous spaces of positive Euler characteristic.

4.1. **The Hirzebruch χ_y-genus.** The Hirzebruch χ_y-genus is defined in [33] by the series

$$f_y(u) = \frac{u(1 + ye^{-u(1+y)})}{1 - e^{-u(1+y)}}.$$

For $y = 0$ it gives the famous Todd genus, while for $y = 1$ it gives the signature.

It is proved in [46] that the Hirzebruch χ_y - genus of a quasitoric manifold can be expressed in terms of combinatorial data of a manifold, that is in terms of the signs and the indexes of the vertices for the simple polytope, which corresponds to the orbit space of the torus action on a manifold.

We proved in [17] the similar result for the Hirzebruch χ_y-genus of homogeneous spaces G/H of positive Euler characteristic related to an arbitrary stable complex structure c_τ equivariant for the canonical action of the maximal torus, under assumption that G/H admits and invariant almost complex structure J. The weights at a fixed point w for the canonical action of the maximal torus T^k on G/H related to c_τ are $\varepsilon_1(\mathrm{w})\mathrm{w}(\alpha_1), \ldots, \varepsilon_n(\mathrm{w})\mathrm{w}(\alpha_n)$, where $\alpha_1, \ldots, \alpha_n$ are the complementary roots for G related to H and $\varepsilon_i(\mathrm{w}) = \pm 1$ for $1 \leq i \leq n$, while the sign of w is given by $\text{sign}(\mathrm{w}) = \varepsilon \cdot \prod_{i=1}^{n} \varepsilon_i(\mathrm{w})$, where $\varepsilon = \pm 1$ depending on whether or not J and c_τ define the same orientation on $\tau(M^{2n})$.

Using Atiyah-Hirzebruch formula and theory of Lie algebras in [17] it is deduced:

Theorem 6. *The Hirzebruch χ_y-genus for $(G/H, c_\tau)$ is given by*

$$\chi_y(G/H, c_\tau) = \varepsilon \cdot \sum_{\mathrm{w} \in W_G/W_H} (-y)^{\frac{1}{2}\sum_{i=1}^{n}(1-\varepsilon_i(\mathrm{w})s_i(\mathrm{w}))} \cdot \prod_{i=1}^{n} \varepsilon_i(\mathrm{w}),$$

where $s_i(\mathrm{w})$ denotes the sign of $\mathrm{w}(\alpha_i)$ regarding to a fix ordering on the canonical coordinates on the Lie algebra $\mathfrak{t}^k$ for T^k.

By putting $y = 0$ and $y = 1$ we obtain the Todd genus and the signature for $(G/H, c_\tau)$ respectively:

Corollary 3.

$$Td(G/H, c_\tau) = \varepsilon \cdot \sum_{\mathrm{w} \in W_G/W_H,\ \varepsilon_i(\mathrm{w}) = s_i(\mathrm{w}),\ 1 \le i \le n} \cdot \prod_{i=1}^{n} \varepsilon_i(\mathrm{w})$$

if the set $\{\mathrm{w} \in W_g/W_H,\ \varepsilon_i(\mathrm{w}) = s_i(\mathrm{w}),$ for all $1 \le i \le n\}$ is nonempty. If this set is empty then

$$Td(G/H, c_\tau) = 0.$$

$$sign(G/H, c_\tau) = \varepsilon \cdot \sum_{\mathrm{w} \in W_G/W_H} (-1)^{\frac{1}{2}\sum_{i=1}^{n}(1-\varepsilon_i(\mathrm{w})s_i(\mathrm{w}))} \cdot \prod_{i=1}^{n} \varepsilon_i(\mathrm{w}).$$

If c_τ is an invariant almost complex structure J we have that $\varepsilon = 1$ and $\varepsilon_i = 1$ for any $1 \le i \le n$, which according to [17] implies:

Corollary 4.

$$\chi_y(G/H, J) = \sum_{\mathrm{w} \in W_G/W_H} (-y)^{\frac{1}{2}\sum_{i=1}^{n}(1-s_i(\mathrm{w}))},$$

$$sign(G/H) = \sum_{\mathrm{w} \in W_G/W_H} (-1)^{\frac{1}{2}\sum_{i=1}^{n}(1-s_i(\mathrm{w}))}.$$

In addition, using the root theory of Lie algebras in [17] we proved:

Corollary 5. *If J is integrable then $Td(G/H, J) = 1$, while for J non-integrable $Td(G/H, J) = 0$.*

4.2. **Rigidity of Hirzebruch genera.** Given an action of the torus T^k on a stable complex manifold the well known question of rigidity immediately arises. A genus $\mathcal{L}_f$ is said to be T^k- rigid if $\mathcal{L}_f^{T^k}(M) = \mathcal{L}_f(M)$. We note that it can be analogously defined the notion of G-rigidity of a genus for manifolds with G-action in the sense that $\mathcal{L}_f$ is G-rigid if an equivariant genus $\mathcal{L}_f^G$ is independent of G, that is $\mathcal{L}_f^G(M) = \mathcal{L}_f(M)$ for any G-manifold M. A genus is said to be rigid if is it G-rigid for any compact connected group G. It is well known [2], [40] that if a genus is S^1-rigid then it is rigid, so also in the case of T^k-action it is enough to consider just S^1-rigidity.

In the case when T^k-action is with isolated fixed points, it is proved in [13] that the conditions for T^k-rigidity of a Hirzebruch genus $\mathcal{L}_f$ can be formulated in terms of functional equations on signs and weights at fixed points.

Theorem 7. *A genus $\mathcal{L}_f$, where f is a series over a $\mathbb{Q}$-algebra A, is T^k-rigid if and only if the functional equation*

$$\sum_{x\in Fix(M)} sing(x)\prod_{j=1}^{n}\frac{1}{f(\omega_j(x)\cdot\mathbf{u})} = c,$$

is satisfied in $A[[u_1,\ldots,u_k]]$ for the constant $c = \mathcal{L}_f(M)$. Here $\omega_j(x)$ are the weight vectors for T^k action related to c_τ at a fixed point x.

4.3. **T^k-rigidity of some important genera on compact homogeneous spaces.** We consider compact homogeneous spaces G/H of positive Euler characteristic. As we already pointed the weight vectors at the identity $e = eH$ for an arbitrary stable complex structure are, up to sign, given by the complementary roots for G related to H, while the weight at the other fixed point we obtain by the action of the quotient W_G/W_H on the weights at the identity, where W_H, W_H are the Weyl groups for H, G respectively. This leads to the possibility of explicit application of Theorem 7. We shortly present here the results from [17] on rigidity of some important Hirzebruch genera on homogeneous spaces under consideration.

4.3.1. *Rigidity of Krichever genus.* Krichever genus was introduced in [41] and it is also known as generalized elliptic genus. It is a Hirzebruch defined by the power series

$$f(u) = \frac{\exp(\mu u)}{B(u,v)},$$

where $B(u,v)$ is Baker-Akhiezer function defined by

$$B(u,v) = B(u,v;\omega_1,\omega_2) = \frac{\sigma(u-v)}{\sigma(u)\sigma(v)}e^{\zeta(v)u}.$$

Here $\sigma(u)$ and $\zeta(u)$ are the Weierstrass sigma and zeta functions and ω_1, ω_2 are half-periods of an elliptic curve Γ such that $\mathrm{Im}\frac{\omega_2}{\omega_1} > 0$.

It is proved in [41] that the Krichever genus is S^1-rigid on the class of SU-manifolds with an equivariant circle action, where these are defined by the condition that their first Chern class vanishes. We reproved in [17] this result for the class of SU-homogeneous spaces of positive Euler characteristic appealing just on Theorem 7 and Lie representation theory. More precisely, we proved the following statement, which together with Liouville theorem [48] proves the rigidity of Krichever genus in our case.

Theorem 8. *If f is the power series which defines Krichever genus, then for any homogeneous space G/H of positive Euler characteristic the function*

$$\sum_{\mathrm{w}\in W_G/W_H} \prod_{j=1}^{n} \frac{1}{f(\mathrm{w}(\alpha_j \cdot u))},$$

has no poles, where α_j are the roots of an arbitrary invariant almost complex SU-structure on G/H.

As examples and remarks in [17] point, the assumption on invariant almost complex structure to be a SU-structure is essential even in the class of homogeneous spaces we consider.

4.3.2. *Rigidity of the elliptic genus of level N.* Appealing on [41] the elliptic genus of level N can be defined as follows. On a elliptic curve Γ fix the points v_{sm} of order N:

$$v_{sm} = \frac{2s}{N}\omega_1 + \frac{2m}{N}\omega_2, \quad s, m = 0, 1, \ldots, N-1$$

and for $\eta_l = \zeta(\omega_l)$, $l = 1, 2$ put

$$\mu_{sm} = -\frac{2s}{N}\eta_1 - \frac{2m}{N}\eta_2 + \zeta(v_{sm}).$$

The elliptic genus of level N is a Hirzebruch genus defined by the series

$$f_{sm}(u) = \frac{\exp(\mu_{sm}u)}{B(u, v_{sm})}.$$

The elliptic genus of level N is proved [32] to be S^1-rigid on the class of S^1-equivariant stable complex manifolds whose first Chern class is divisible by N. In [17] we reproved this results for compact homogeneous spaces of positive Euler characteristics appealing just on Lie representation theory and Theorem 7.

Theorem 9. *The elliptic genus of level N is T^k -rigid on homogeneous spaces of positive Euler characteristic endowed with the canonical action of the maximal* *torus T^k and with an invariant almost complex structure whose sum of roots is divisible by N.*

4.3.3. *Rigidity of a Hirzebruch genus defined by an odd series.* We consider now Hirzebruch genera defined by an odd power series. These genera include the signature which we already mentioned and which coincides with the L-genus defined by an odd series $f(u) = \tanh(u)$, as well as another important example such as $\hat{A}$-genus which is defined by an odd series $f(u) = 2\sinh(\frac{u}{2})$. More generally, an elliptic genus is defined by an odd power series $f(u) = g^{-1}(x)$ for $g(x) = \int_0^x \frac{dt}{\sqrt{1-2\delta t^2+\varepsilon t^4}}$, where the coefficient ring $R = \mathbb{C}$ and $\delta^2 \neq \varepsilon \neq 0$. For $\varepsilon = \delta = 0$ the degenerate elliptic genus gives the signature, while for $\delta = \frac{-1}{8}$ and $\varepsilon = 0$ it gives $\hat{A}$-genus.

We proved in [17] that $\mathcal{L}_f$ for an odd power series f, is T^k-rigid and equal to zero on a large class of homogeneous spaces, being a stronger result than T^k-rigidity. In particular, it holds:

Theorem 10. *Any Hirzebruch genus $\mathcal{L}_f$ defined by an odd series f is*

- *T^n-rigid and equal to zero on $U(n)/T^n$ endowed with an arbitrary T^n–equivariant stable complex structure.*
- *T^{km}-rigid and equal to zero on $U(km)/(U(m))^k$ endowed with an arbitrary T^{km}-equivariant stable complex structure for odd m.*

Theorem 11. *The elliptic genus and the $\hat{A}$-genus are equal to zero on:*

- *$U(n)/T^n$ related to an arbitrary T^n-equivariant stable complex structure.*
- *$U(km)/(U(m))^k$ related to an arbitrary T^{km}-equivariant stable complex structure for odd m.*

5. Some important torus actions

An action of a group G on a set X is a map $\alpha : G \times X \to X$ denoted by $\alpha(g, x) = g \cdot x$ such that $\alpha(g_1 g_2, x) = \alpha(g_1, g_2 \cdot x)$ and $\alpha(e, x) = x$, where e is an identity element in G. For $x \in X$ by $G_x = \{g \in G | g \cdot x = x\}$ is usually denoted its isotropy subgroup and its orbit by $Gx \subseteq X$. The sets G/G_x and Gx correspond bijectively and G_y and G_x are conjugate in G for any $y \in Gx$. It naturally arises projection map $\pi : X \to X/G$, where X/G is the set of disjoint orbits. It may happen, but not obligatory, that this projection has, in a sense, inverse map which is called a section. This is the map $s : X/G \to X$ which is defined by choosing some representative $s(Gx)$ for each orbit, such that $\pi(s(Gx)) = Gx$. The section s defines further a characteristic function $\lambda_s : X/G \to \mathcal{S}(G)$ by $\lambda_s(Gx) = G_{s(Gx)}$, where $\mathcal{S}(G)$ stands for the set of all subgroups in G.

On the other hand, given a characteristic function $\chi : Y \to \mathcal{S}(G)$ on an arbitrary set Y, it is defined the set

$$D(\lambda) = (G \times Y)/\sim,$$

where the equivalence relation is given by $(g_1, y) \sim (g_2, y)$ whenever $g_1^{-1} g_2 \in \lambda(y)$. In this way $D(\lambda)$ is equipped with an action of G by $g \cdot [h, y] = [gh, y]$. The orbits are the subsets $\{[g, y] : g \in G\}$ and the quotient map is the projection on the second factor $\pi : D(\lambda) \to Y$, while the canonical section is given by $s(y) = [1, y]$ for any $y \in Y$.

It can be easily verified that the constructions $D(\lambda)$ and λ_s are inverse to each other, thus they establish a correspondence between characteristic functions and sets with a G-action having a section.

In order to put this into topological context one proceeds by assuming that α is a continuous action of a topological group G on a topological space X. Then each isotropy subgroup is closed, so the characteristic map $\lambda_s : X/G \to \mathcal{S}(G)$ takes values in the poset of closed subgroups and it is continuous with respect to the lower topology on $\mathcal{S}(G)$.

In the case of compact torus action, that is when $G = T$, we will see that, due to existence of a section, the above correspondence holds for toric and quasitoric manifolds. This further leads to the remarkable combinatorial description of algebraic topology of these manifolds. For higher complexity actions, such section does not exist any more and we present the theory of $(2n, k)$-manifolds which develops the tools and techniques for study of such manifolds.

5.1. **Toric manifolds.** Let $\mathbb{C}^*$ denotes the multiplicative group of complex numbers. The product $(\mathbb{C}^*)^n$ of n copies of $\mathbb{C}^*$ is known as the algebraic torus following the line that the compact torus T^n is the product of n copies of the circle. The expression $T^n = \{(e^{2\pi i\varphi_1}, \dots, e^{2\pi i\varphi_n}) \in \mathbb{C}^n, \quad \varphi_1, \dots, \varphi_n \in \mathbb{R}\}$ gives the subgroup embedding of T^n in $(\mathbb{C}^*)^n$.

Def 1. A toric variety over $\mathbb{C}$ is a n-dimensional normal variety X containing $(\mathbb{C}^*)^n$ as a Zariski open set in such a way that the natural action of $(\mathbb{C}^*)^n$ on itself extends to an action of $(\mathbb{C}^*)^n$ on X.

It follows that $(\mathbb{C}^*)^n$ acts on X with a dense orbit. A compact, smooth toric variety are nowadays, in particular in topology, called a toric manifold. In spite of their relatively simple definition toric manifolds have many remarkable features and applications. One of the most beautiful features of a toric manifold is that its many subtle properties can be expressed in the language of combinatorics and convex geometry.

The basic combinatorial object associated to a toric variety is a fan. Let $\mathbb{R}^n$ be the Euclidean space and $\mathbb{Z}^n \subset \mathbb{R}^n$ an integral lattice. A rational convex polyhedral cone σ in $\mathbb{R}^n$ is a cone spanned by the finite set of vectors $l_1, \dots, l_s \in \mathbb{Z}^n$, that is $\sigma = \{r_1 l_1 + \ldots + r_s l_s \in \mathbb{R}^n, \quad r_i \geq 0\}$, which does not contain line through the origin. A fan Σ in $\mathbb{R}^n$ is a non-empty collection of rational convex polyhedral cones in $\mathbb{R}^n$ which is closed under taking faces, that is a face of any cone is again cone in Σ, and it is closed under intersection, that is the intersection of two cones in Σ is a face of each. A fan Σ is said to be complete if the union of all fans from Σ covers entire space $\mathbb{R}^n$, and non-singular if every cone of dimension k in Σ is spanned by k integer vectors which form a part of a basis in $\mathbb{Z}^n$.

A fundamental result in the theory of toric varieties states that the category of toric varieties is equivalent to the category of fans, that is there is a bijective correspondence $X \to \Sigma_X$ see [51], [52]. A toric variety X is compact if and only if its corresponding fan Σ_X is complete, and X is smooth if and only if Σ_X is non-singular. Thus, a toric variety X is a toric manifold if and only if Σ_X is a smooth and non-singular.

Toric varieties can be as well defined from convex polytopes. Recall that a convex polytope P is a bounded convex polyhedron, where a polyhedron is defined by the intersection of finitely many half-spaces in some $\mathbb{R}^n$, that is $P = \{\mathbf{x} \in \mathbb{R}^n \ : \langle l_i, \mathbf{x}\rangle \geq a_i, \ i = 1, \dots, m\}$. where $l_i \in (\mathbb{R}^n)^*$ are some linear function and $a_i \in \mathbb{R}$, $1 \leq i \leq m$. Assume we are given a convex n-polytope P^n with vertices in integer lattice $\mathbb{Z}^n \subset \mathbb{R}^n$. Then the vectors l_i can be chosen integer and primitive, and the numbers a_i can be chosen integer. In this case l_i is normal to the facet $F_i \subset P^n$ and is pointing inside the polytope P^n. Define the complete fan $\Sigma(P)$ whose cones are generated by those sets of normal vectors $l_{i_1}, \dots, l_{i_k}$ whose corresponding facets $F_{i_1}, \dots, F_{i_k}$ have non-empty intersection in P^n . This fan is called the normal fan of P^n.

Consider the toric variety $M_{\Sigma(P)}$ corresponding to the fan $\Sigma(P)$ and define $M_P = M_{\Sigma(P)}$. The underlying topological space of the toric variety M_P can be identified with the following quotient space:

$$M_P = T^n \times P^n/\!\sim,$$

where $(t_1, p)\sim(t_2, q)$ if and only if $p = q$ and $t_1 t_2^{-1} \in T(q)$.

The orbit space M_P/T^n identifies with P^n and $T(q)$ is a subtorus in T^n which is given by the stabilizer of the points in M_P which project to the point q. Note that if q is a vertex then $T(q) = T^n$, while $T^n = \{e\}$ if q belongs to the interior of P^n.

Ex 1. Consider the complex projective space $\mathbb{C}P^n = \{(z_0 : \ldots : z_n), \ z_i \in \mathbb{C}\}$ with the linear action of $(\mathbb{C}^*)^n$ given by $(t_1, \dots, t_n) \cdot (z_0 : \ldots : z_n) = (z_0 : t_1 z_1 : \ldots : t_n z_n)$. It is obvious that $(\mathbb{C}^*)^n \subset \mathbb{C}P^n$ is a dense open subset. A fan defining $\mathbb{C}P^n$

consists of the cones spanned by all proper subsets of the vectors $e_1, \dots, e_n, -e_1 - \dots - e_n$ in $\mathbb{R}^n$. The toric manifold $\mathbb{C}P^n$ arises from the polytope $P = \Delta^n$, which is the standard n-simplex in $\mathbb{R}^n$, that is $\mathbb{C}P^n = X_{\Delta^n}$.

A fan Σ defines the simplicial complex K_Σ whose vertex set is $\{1, \dots, m\}$ and $\{i_1, \dots, i_k\} \subset \{1, \dots, m\}$ is a simplex in K_Σ if and only if $l_{i_1}, \dots, l_{i_k}$ span a cone of Σ. Now, write the primitive integer vectors along the rays of Σ in the standard basis of $\mathbb{Z}^n$, that is $\mathbf{l}_i = (l_{1i}, \dots, l_{ni})$, $i = 1, \dots, m$ and assign to each vector $\mathbf{l}_i$ an independent variable v_i. The Stanley - Reisner ring of a simplicial complex K_Σ is the quotient ring

$$R(K_\Sigma) = R[v_1, \dots, v_m]/\mathcal{I}_{K_\Sigma},$$

where $\mathcal{I}_{K_\Sigma}$ is the homogeneous ideal generated by all free-square monomials $v_\sigma = v_{i_1} v_{i_2} \cdots v_{i_s}$, $i_1 < i_2 \dots < i_s$ such that $\sigma = \{i_1, \dots, i_s\}$ is not a simplex of K_Σ.

Define the linear forms

$$\theta_i = l_{i1} v_1 + \dots + l_{im} v_m \in \mathbb{Z}[v_1, \dots, v_m],\ 1 \le i \le n.$$

Denote by $\mathcal{J}_\Sigma$ the ideal in $\mathbb{Z}[v_1, \dots, v_m]$ spanned by these linear forms. The following was proved in [23], [35]:

Theorem 12. *The integral cohomology ring of M_Σ is given by*

$$H^*(M_\Sigma . \mathbb{Z}) = \mathbb{Z}[v_1, \dots, v_m]/(\mathcal{I}_{K_\Sigma} + \mathcal{J}_\Sigma),$$

where $\dim v_i = 2$ *for* $1 \le i \le m$.

5.2. **Quasitoric manifolds.** A quasitoric manifold is, in a sense, a topological counterpart to a toric manifold in algebraic geometry, that is a topological analogue of a nonsingular projective toric variety of algebraic geometry. The notion of quasitoric manifolds was introduced by Davis-Januskiewitz in [24], while for detail elaboration with updates on current development we refer to the books by Buchstaber-Panov [11], [12].

A quasitoric manifold is a closed smooth manifold M^{2n} of real dimension $2n$ with a smooth action of the compact torus T^n such that

(1) T^n-action is locally standard, that is M^{2n} has a standard atlas, meaning that any chart is given by a triple $(U_\alpha, u_\alpha, \varphi_\alpha)$, where U_α is an open T^n-invariant subset in M^{2n}, φ_α is an automorphism of T^n and $u_\alpha : U_\alpha \to V_\alpha \subset \mathbb{C}^n$ is an φ_α-equivariant homeomorphism for some standardly T^n-invariant set W_α, that is $u_\alpha(\mathbf{t} \cdot x) = \varphi_\alpha(\mathbf{t}) \cdot u_\alpha(x)$ for any $\mathbf{t} \in T^n$ and $x \in U_\alpha$.
(2) there is a continuous projection map $\mu : M^{2n} \to P^n$, where P^n is a simple convex polytope, whose fibers are T^n-orbits.

Property (1) implies that the orbit space M^{2n}/T^n is a manifold with corners, and property (2) implies that this orbit space is diffeomorphic, as a manifold with corners, to the simple polytope P^n. The definition also implies that μ maps every p-dimensional orbit to a point in the relative interior $\overset{\circ}{F}$ of a p-dimensional face F of P^n and that the preimage $\mu^{-1}(x)$ is a p-dimensional orbit for any $x \in \overset{\circ}{F}$. Thus, the T^n - action is free on $\mu^{-1}(\overset{\circ}{P^n})$, while the vertices correspond by μ to the fixed point of T^n-action on M^{2n}. Moreover, for any face F of P^n all points from $M_F = \mu^{-1}(\overset{\circ}{F})$ have the same stabilizer. The projection map μ is commonly called (almost) moment map, following the notation from algebraic and symplectic geometry.

Ex 2. The complex projective space $\mathbb{C}P^n$ with the action of T^n given in Example 1 is a quasitoric manifold over the simplex Δ^n. The moment map $\mu : \mathbb{C}P^n \to \Delta^n$ is given by

$$(z_0 : \ldots : z_n) \to \frac{1}{\sum\limits_{i=0}^{n} |z_i|^2}(|z_0|^2, \ldots, |z_n|^2).$$

Remark 2. Projective toric manifolds provide examples of quasitoric manifolds, while there are quasitoric manifolds which are not toric manifolds. For example, the connected sum $\mathbb{C}P^2 \# \mathbb{C}P^2$ is a quasitoric manifold with an appropriate action of T^2, but it is not a toric manifold because it does not admit any almost complex structure.

A quasitoric manifold can be completely recovered using the combinatorics of the polytope P^n and the characteristic function. We recall this beautiful construction. Let $\{F_1, \ldots, F_m\}$ be the set of all facets of P^n. The stationary subgroups $T(F_i)$ of these facets are one-dimensional tori in T^n, so they can be represented by $T(F_i) = \{(e^{2\pi i\lambda_{1i}\phi}, \ldots, e^{2\pi i\lambda_{ni}\phi}),\ \varphi \in \mathbb{R}\}$ for a fixed vector $\lambda_i = (\lambda_{1i}, \ldots, \lambda_{ni}) \in \mathbb{Z}^n$. The vectors λ_i, $1 \le i \le m$ satisfy important condition: if intersection $F_{i_1} \cap \ldots \cap F_{i_n}$ is a vertex of P^n, then the vectors $\lambda_{i_1}, \ldots, \lambda_{i_n}$ form a basis in $\mathbb{Z}^n$.

The characteristic map $l : \{F_i\} \to S(T^n)$, where $S(T^n)$ is the set of all connected subgroups of the torus T^n defined by $l(F_i) = T(F_i)$, can be described using characteristic matrix Λ whose columns are the vectors λ_i, $1 \le i \le m$. It is the result of Davis-Januszkiewicz [24] that the matrix Λ together with the combinatorics of the polytope P^n determine the cohomology of M^{2n}.

Let $\mathbf{F}$ denotes the partially ordered set of all faces of P^n. The points from $\mu^{-1}(F)$, for any $F \in \mathbf{F}$ have the same stabilizer. It implies that the characteristic map l extends to the map $l : \mathbf{F} \to S(T^n)$, which to each $F = F_{i_1} \cap \ldots \cap F_{i_k} \in \mathbf{F}$ assigns the stationary subgroup of the set $\mu^{-1}(F)$, that is $T(F_{i_1}) \times \ldots \times T(F_{i_k}) \subset T^n$.

The map $l : \mathbf{F} \to S(T^n)$ is completely determined by the matrix Λ. A quasitoric manifold can be recovered, up to diffeomorphism, using the characteristic pair (P^n, Λ), that is

$$M^{2n} \cong (T^n \times P^n)/\sim,$$

where $(t_1, p)\sim(t_2, q)$ if and only if $p = q$ and $t_1 t_2^{-1} \in \Lambda(F(p))$, where $F(p)$ is the smallest face of the polytope P^n which contains p. This is proved in [24], using the observation that for a quasitoric manifold M^{2n} there is the section $s : P^n \to M^{2n}$ defined by $s(x) = 1 \cdot x$ and that P^n is a contractible topological space.

It is proved in [24] that the cohomology ring of a quasitoric manifold has the same structure as the cohomology ring of a nonsingular compact toric variety. Hence, it is generated by the two-dimensional classes classes which satisfy two type of relations: monomial relations coming from the face ring of the corresponding simple polytope P and linear relations coming from the characteristic matrix.

Theorem 13. (1) *The integral homology groups of a quasitoric manifold $M^{2n} = (P^n, \Lambda)$ vanish in odd dimensions, so they are free abelian groups in even dimensions. Their Betti numbers are given by*

$$b_{2i}(M) = h_i(P^n),$$

where $h_i(P^n)$, $i = 0, \ldots,$ are the components of the h-vector of P^n.

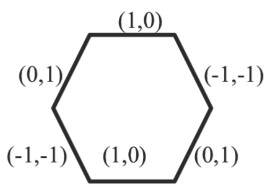
(2) *The integral cohomology ring of M^{2n} is given by*

$$H^*(M^{2n}, \mathbb{Z}) = \mathbb{Z}[v_1, \ldots, v_m]/\mathcal{I},$$

where v_i are two-dimensional cohomology classes and $\mathcal{I}$ is the ideal generated by the elements of the following two types:

- $v_{i_1} \cdots v_{i_k}$ *whenever $F_{i_1} \cap \cdots \cap F_{i_k} = \emptyset$ – the Stanley-Reisner relations;*
- *the linear forms $t_i = \lambda_{i1} v_1 + \ldots + \lambda_{im} v_m$, $1 \leq i \leq n$.*

Ex 3. Let us consider the quasitoric manifold M^4 over the hexagon, whose characteristic matrix is given on the picture by assigning to each edge the corresponding vector:



According to Theorem 13 we obtain

$$H^*(M^4, \mathbb{Z}) = \mathbb{Z}[v_1, \ldots, v_6]/\mathcal{I},$$

where $\mathcal{I}$ is generated by the polynomials:

- $v_1v_3, v_1v_4, v_1v_5, v_2v_4, v_2v_5, v_2v_6, v_3v_5, v_3v_6, v_4v_6,$
- $-v_1 + v_2 - v_4 + v_5, -v_1 + v_3 - v_4 + v_6.$

5.3. **Torus actions of complexity one.** In the previous cases of toric and quasitoric manifolds we considered nicely behaved effective actions of the torus T^n on a manifold M^{2n} of the dimension $2n$, that is the dimension of the acting torus is equal half of the dimension of a manifold. It is naturally to try to generalize this and to consider similar effective actions of the torus T^k on M^{2n} for any $k \leq n$. It arises to be important here the number $d = n - k$ which is commonly called the complexity of the action and it generalizes the notion of complexity of algebraic torus $(\mathbb{C}^*)^k$ - action from algebraic geometry and symplectic geometry. While, as we demonstrated, the actions of complexity zero are well studied establishing remarkable combinatorial results, it turns out that the study of torus actions of the complexity $d \geq 1$ is much harder problem. The study of actions of complexity $d \geq 2$ are generally assumed in the literature to be quite difficult problem. Still, actions of complexity one have, in a sense, intermediate positions, since there already have been studied from different point of views and some nice results have been achieved.

In algebraic geometry the actions of complexity one of general algebraic groups are studied by many authors [51], [2], [31], [30]. In particular, the classification of such an actions is given by Timashëv in [52]

Symplectic toric manifolds are known to be classified by their moment map image [25]. Complexity one symplectic torus actions on symplectic manifolds, and among them the Hamiltonian ones, are also well studied, related results can be found in the series of papers by Karshon-Tolman, [37], [38], [39]. In particular, they proved that if the action of the torus T^{n-1} on a compact connected symplectic manifold M^{2n} is of complexity one, then there exists a connected closed oriented surface Σ such that the orbit space M^{2n}/T is homeomorphic to the quotient of the space $\Delta \times \Sigma$ by some appropriate equivalence relation, where Δ is the corresponding moment polytope. Moreover, if T^{n-1}-action on M^{2n} has isolated fixed points then Σ is the sphere S^2. Recall that a complexity one symplectic torus action on a compact symplectic manifold is Hamiltonian if and only if it has a fixed point, so the orbit space of a Hamiltonian symplectic manifold M^{2n} by the complexity one torus action is of the form

$$M^{2n}/T \cong (\Delta \times S^2)/\sim,$$

by some appropriate relation $\sim$, see [39]. In particular, if a complexity one action is Hamiltonian and in general position, they proved that M^{2n}/T^{n-1} is homeomorphic to the sphere S^{n+1}.

An independent study of complexity one torus actions from the point of view of topology has been done by Ayzenberg [4], [5], [6]. For example, the orbit space

M^{2n}/T^{n-1} of a smooth, closed, connected, orientable manifold by complexity one torus action in general position, is proved to be a closed topological manifold. If one omits the condition that an action is in general position, then the orbit space is a closed topological manifold with corners.

Ex 4. Some well studied homogeneous examples of complexity one canonical torus actions are:

- T^4-action on the complex Grassmann manifold $G_{4,2} = U(4)/U(2)\times U(2)$, [38], [19];
- T^n-action on the complete complex flag manifold $F_3 = U(3)/T^3$, [20], [4];
- T^3-action on the quaternionic projective plane $\mathbb{H}P^2 = Sp(3)/Sp(1) \times Sp(2)$, [4];
- T^2-action on the sphere $S^6 = G_2/U(3)$, [4].

In particular, it is proved in [4] that $\mathbb{H}P^2/T^3 \cong S^5$ and that $S^6/T^2 \cong S^4$.

6. Theory of $(2n, k)$-manifolds

As we saw in the previous sections, a toric or quasitoric manifold is modeled on the product $T^n \times P^n$, meaning that to any point of such a manifold it can be assigned two coordinates (t, x), that is an angle coordinate and moment map coordinate. It turns out that this is very distinctive feature of these manifolds with zero complexity torus action.

If we go wider and consider for example very important class of manifolds such as homogeneous spaces G/H , then the canonical action of the compact torus T of G on $M = G/H$ has, in general, complexity $d \geq 1$ and, intuitively, it is not difficult to observe that such two-coordinate description of points from G/H, even locally, is not possible. This is also due to the observation that in this case the orbit space M/T is not a polytope any more, in fact it turns out its description to be a hard problem. Moreover, it can be proved, that even for some fundamental homogeneous spaces, such as Grassmann manifolds, a section from the corresponding moment map polytope to the initial space does not exist.

In the series of papers by Buchtaber-Terzić, the problem of smooth, effective T^k -action on a smooth, closed, oriented manifold M^{2n} of an arbitrary complexity $d \geq 0$ has been studied. This study started with the papers [18], [19] where the canonical action of the torus T^n on a complex Grassmann manifold $G_{n,k} = U(n)/U(k) \times U(n-k)$ is studied. In particular, the corresponding orbit spaces are explicitly described for $n = 4$ and $n = 5$. Afterwards, the general theory of $(2n, k)$-manifolds has been developed in [20]. Then, in sequent papers [21] and [22], a very explicit model for the orbit space $G_{n,2}/T^n$ was constructed and it was established a beautiful connection between this model and the well known construction $G_{n,2}/\!/(\mathbb{C}^*)^n$ from algebraic geometry known as the Chow quotient.

We shortly present the results of these papers in the remaining part of the text, referring to them for all necessary notions and proofs.

The motivation to study the canonical T^n - action on a complex Grassmann manifold $G_{n,k}$ came to us from the papers of Gel'fand-Serganova [29] and Goresky-MacPherson [28] in which some fundamental results on $(\mathbb{C}^*)^n$-action on $G_{n,k}$ are proved.

We recall first that the standard moment map $\mu_{n,k} : G_{n,k} \to \Delta_{n,k} \subset \mathbb{R}^n$ is defined by

$$\mu_{n,k}(L) = \frac{1}{\sum |P^J(L)|^2} \sum |P^J(L)|^2 \Lambda_J,$$

where $P^J(L)$ are the Plücker coordinates for L, $\Lambda_J \in \mathbb{R}^n$, $(\Lambda_J)_i = 1$ for $i \in J$, while $(\Lambda_J)_i = 0$ for $i \notin J$ and $J \subset \{1, \ldots, n\}$, $\|J\| = k$. Its image is a convex $(n-1)$-polytope, which is the convex hull over all vertices Λ_J, so it belongs to the hyperplane $x_1 + \ldots + x_n = k$. This polytope is known as the hypersimplex and it is denoted by $\Delta_{n,k}$.

The moment map μ is T^n-invariant, where $\mathbb{R}^n$ is endowed with the trivial T^n-action, meaning that it induces the map $\hat{\mu} : G_{n,k}/T^n \to \Delta_{n,k}$.

On the other hand the Plücker coordinates define an atlas on $G_{n,k}$ whose charts are defined by

$$M_J = \{L \in G_{n,k} : P^J(L) \neq 0\}, \ \ J \subset \{1, \ldots, n\}, \ \|J\| = k.$$

We denote by $Y_J = G_{n,k} \backslash M_J$. The strata W_σ on $G_{n,k}$, $\sigma \subset \{J \subset \{1, \ldots, n\}, \ |J| = k\}$ we define to be non-empty subsets of the form:

$$W_\sigma = (\bigcap_{J \in \sigma} M_J) \cap (\bigcap_{J \notin \sigma} Y_J). \tag{3}$$

The strata are T^n-invariant, pairwise disjoint and their union is entire $G_{n,k}$, that is they give a stratification of $G_{n,k}$ in a classical sense. In [19] we proved:

Theorem 14. *For a stratum W_σ it holds:*

- $\mu(W_\sigma) = \overset{\circ}{P}_\sigma$, *that is an interior of the polytope which is obtained as the convex hull over the vertices Λ_J, $J \in \sigma$,*
- *The map $\hat{\mu} : W_\sigma/T^n \to \overset{\circ}{P}_\sigma$ is a fiber bundle with a fiber F_σ being an algebraic manifold.*

Remark 3. Note that the strata defined in this way coincide with the strata as defined by Gel'fand-Serganova in [29].

A polytope $P \subset \Delta_{n,k}$ such that $P = P_\sigma$ for some stratum W_σ is called an admissible polytope. We denote by F_σ the homeomorphic type of a fiber space for the fiber bundle $\hat{\mu} : W_\sigma/T^n \to \overset{\circ}{P}_\sigma$ and call it the space of parameters of the stratum W_σ.

Ex 5. The set of admissible polytopes of a quasitoric manifold consists of a simple polytope P and is faces. In addition, the space of parameters for any strata is a point.

There are relatively a lot strata on any Grassmannian $G_{n,k}$ whose space of parameters is not a point, as even low-dimensional examples $G_{4,2}$ and $G_{5,2}$ demonstrate. This means that even the orbit space of a stratum is not homeomorphic to a polytope in general, that is the parameter coordinate in the orbit space of a stratum arises as well.

Since $\overset{\circ}{P}_\sigma$ is a contractible space, from Theorem 14 it follows

$$W_\sigma/T^n \cong \overset{\circ}{P}_\sigma \times F_\sigma.$$

In this way we obtain that

$$G_{n,k}/T^n = \bigcup_\sigma W_\sigma/T^n = \bigcup_\sigma \overset{\circ}{P}_\sigma \times F_\sigma. \tag{4}$$

In order to describe the orbit space $G_{n,k}/T^n$ one should find an ambient space for the pieces $\overset{\circ}{P}_\sigma \times F_\sigma$ by which it would be possible to describe their gluing, that is to describe the topology on their union which will give the orbit space $G_{n,k}/T^n$. In resolving this problem the main stratum has an extremely important role.

The main stratum W is defined as the intersection of all charts, that is $W = \bigcap_J M_J$. It is an open, dense set in $G_{n,k}$, so its compactification gives entire Grassmannian $G_{n,k}$. It implies that $W/T^n \cong \overset{\circ}{\Delta}_{n,k} \times F$ is a dense set in $G_{n,k}/T^n$, where F is the space of parameters of W. Therefore, the above mentioned problem of finding an ambient space for topological realization of union (4) is equivalent to find, in a sense, a suitable compactification for $\overset{\circ}{\Delta}_{n,k} \times F$. This problem naturally reduces to finding a suitable compactification for F. For such compactification $\mathcal{F}$ for F we said to be a universal space of parameters for the T^n - action on $G_{n,k}$. Before we proceed with formal definitions and constructions let us formulate some results obtained for the Grassmannians $G_{4,2}$ and $G_{5,2}$ using the presented approach, see [18], [19].

Theorem 15. *The universal space of parameters for $G_{4,2}$ is $\mathbb{C}P^1 \cong S^2$ and the orbit space $G_{4,2}/T^4$ is homeomorphic to $\Delta_{4,2} \times \mathbb{C}P^1$ quotiened by the relation $(x, c_1) \sim (y, c_2)$ if and only if $x = y \in \partial\Delta_{4,2}$, that is*

$$G_{4,2}/T^4 \cong S^4.$$

Theorem 16. *The universal space of parameters for $G_{5,2}$ is the del Pezzo surface of degree 5, which is obtained as the blow up of the surface $F = \{((c_1 : c_1^{'}), (c_2 : c_2^{'}), (c_3 : c_3^{'})) \in (\mathbb{C}P^1)^3 | c_1 c_2^{'} c_3 = c_1^{'} c_2 c_3^{1}\}$ at the point $((1 : 1), (1 : 1), (1 : 1))$. The orbit space $G_{5,2}/T^5$ is homotopy equivalent to the space which is obtained by attaching the disc D^8 to the four suspension $\Sigma^4 \mathbb{R}P^2$ by the generator of the homotopy group $\pi_7(\Sigma^4 \mathbb{R}P^2)$.*

Generalizing the notions and construction for the Grassmann manifolds, it is developed in [20] the theory of $(2n, k)$-manifolds. The structural data for such a manifold are specified through the system of axioms and, afterwards, it is obtained the model for the orbit space M^{2n}/T^k in terms of these structural data.

Let M^{2n} be a smooth, closed, oriented manifold M^{2n} endowed with a smooth, effective action θ of the torus T^k, $1 \leq k \leq n$, such that the stabilizer of any point is a connected subgroup of T^k. In addition, it is assumed an existence of a smooth θ-equivariant map $\mu : M^{2n} \to \mathbb{R}^k$, whose image is a k-dimensional convex polytope P^k. The map μ is called the almost moment map of the given T^k-action on M^{2n}.

The triple (M^{2n}, θ, μ) is called a $(2n, k)$-manifold, if it satisfies the following set of axioms.

Axiom 1. There is a smooth atlas $\mathcal{A} = \{(M_i, \alpha_i)\}_{i \in I}$, where M_i is an open, dense, T^k-invariant subset in M^{2n}, which contains exactly one fixed point x_i with $\alpha_i(x_i) = (0, \ldots, 0)$ and $x_i \neq x_j$ for $i \neq j$.

Using this axiom one can, at this step, define the strata W_σ by (14).

Axiom 2. The almost moment map μ is a bijection between the set of T^k-fixed points $\{x_i\}$ and the set of vertices of the polytope P^k.

Axiom 3. The characteristic function $\chi : M^{2n} \to S(T^k)$, which to each point $x \in M^{2n}$ assigns its stabilizer $T_x \subset T^k$ regarding to the given T^k-action, is constant on any stratum W_σ, that is $T_x = T_\sigma$ for any $x \in W_\sigma$.

Axiom 4. The almost moment map μ maps a stratum W_σ onto $\overset{\circ}{P}_\sigma$, where $P_\sigma = \text{convhull}\{\Lambda_J, J \in \sigma\}$ and it induces the fiber bundle $\hat{\mu}_\sigma : W_\sigma/T^\sigma \to \overset{\circ}{P}_\sigma$, where $\dim P_\sigma = \dim T^\sigma$ and $T^\sigma = T^k/T_\sigma$.

A polytope $P \subset P^k$ whose interior can be obtained as the image of a stratum by the moment map, is referred to as an admissible polytope. Since $P = P_\sigma$ is a contractible space, this axiom implies that there exists a trivialization, that is a homeomorphism $h_\sigma : W_\sigma/T^\sigma \to \overset{\circ}{P}_\sigma \times F_\sigma$. It induces the projection $\xi_\sigma : W_\sigma/T^\sigma \to F_\sigma$. Thus, for any $c_\sigma \in F_\sigma$, we can define the subspace $W_\sigma[\xi_\sigma, c_\sigma]$ of W_σ called the leaf, by

$$W_\sigma[\xi_\sigma, c_\sigma] = (\pi_\sigma^{-1} \circ \xi_\sigma^{-1})(c_\sigma),$$

where $\pi_\sigma : W_\sigma \to W_\sigma/T^\sigma$ is a projection.

Axiom 5. Any leaf $W_\sigma[\xi_\sigma, c_\sigma]$ in W_σ is a smooth submanifold in M^{2n} and the induced map $\mu_{\xi_\sigma, c_\sigma} : W_\sigma[\xi_\sigma, c_\sigma] \to \overset{\circ}{P}_\sigma$ is a smooth fiber bundle, then the boundary $\partial W_\sigma[\xi_\sigma, c_\sigma]$ is the union of leafs $W_{\bar\sigma}[\xi_{\bar\sigma}, c_{\bar\sigma}]$ for exactly one $c_{\bar\sigma} \in F_{\bar\sigma}$, where $P_{\bar\sigma}$ runs through some admissible faces of P_σ and the map $\eta_{\sigma,\bar\sigma} : F_\sigma \to F_{\bar\sigma}$ obtained in this way is a continuous map.

Denote by $C(M^{2n}, P)$ the complex of admissible polytopes, meaning that we consider the formal union of admissible polytopes. We have canonical projection $\hat\pi : M^{2n} \to C(M^{2n}, P^k)$ and it can be proved that there exists the canonical map $f : M^{2n} \to C(M^{2n}, P^k)$ such that $\mu = \hat\pi \circ f$. A topology on $C(M^{2n}, P^k)$ is defined to be the quotient topology related to the map f. Now, consider the space

$$\mathfrak{E}(M^{2n}, T^k) = \{(x, y) \in C(M^{2n}, P^k) \times M^{2n} : x \in P_\sigma, y \in W_\sigma, \hat\pi(x) = \mu(y)\}.$$

It can be proved that $\mathfrak{E}(M^{2n}, T^k)$ is a compact Hausdorff topological space, which is homeomorphic to M^{2n}. This implies that the orbit space $\mathfrak{E}(M^{2n}, T^k)/T^k$ regarding to the naturally defined torus action is also compact Hausdorff topological space homeomorphic to M^{2n}/T^n.

Axiom 6. There exists a compactification $\mathcal{F}$ for F called the universal space of parameters and there exist topological spaces $\tilde F_\sigma$ called virtual spaces of parameters such that:

(1) $\tilde F = F$, $\bigcup_\sigma \tilde F_\sigma = \mathcal{F}$ and $\tilde F_\sigma \subset \tilde F_{\bar\sigma}$ for any $\bar\sigma \subset \sigma$ such that $P_{\bar\sigma}$ is a face of P_σ;

(2) there is a continuous projection $p_\sigma : \tilde F_\sigma \to F_\sigma$ such that $p_{\bar\sigma} \circ i_{\sigma,\bar\sigma} = \eta_{\sigma,\bar\sigma} \circ p_\sigma$ for any σ, where $i_{\sigma,\bar\sigma}$ is an inclusion given by the previous item.

(3) the map $H : \mathcal{E} = \bigcup_\sigma P_\sigma \times \tilde F_\sigma \to \mathfrak{E}(M^{2n}, T^k)/T^n$ defined by $H(x_\sigma, \tilde c_\sigma) = (x_\sigma, p_\sigma(\tilde c_\sigma))$ is a continuous map, where the topology on $\mathcal{E}$ is induced by the embedding $\mathcal{E} \to C(M^{2n}, P^k) \times \mathcal{F}$.

Using these axioms, a model for the orbit space M^{2n}/T^k is constructed in [20] and it gives an effective tool for the description of M^{2n}/T^k in terms of the three fundamental ingredients consisting of moment map coordinate, toral coordinate and parameter coordinate. Our main statement can be now formulated as follows:

Theorem 17. *The quotient space of the space $\mathcal{E}$ by the map H is homeomorphic to the orbit space M^{2n}/T^k.*

The theory of $(2n, k)$-manifolds covers all previously considered manifolds and much more. In [20] it is proved that the following manifolds satisfy Axioms 1-6.

- Quasitoric manifolds M^{2n} are $(2n, n)$-manifolds;
- Complex Grassmann manifolds $G_{n,k}$ are $(2k(n-k), n-1)$-manifolds;

- The full complex flag manifolds F_n are $(n(n-1), n-1)$-manifolds. In particular, $F_3/T^3 \cong S^4$.

The results of the papers [29], [28] and [36] suggest that the Grassmann manifolds $G_{n,2}$ should be additionally studied taking into account their specific. This specific firstly shows up through the fact that the strata for $G_{n,2}$ glue nicely together, that is the boundary of a stratum is the union of strata. For the other Grassmann manifolds this is not the case as Gel'fang-Serganova example shows [29], [20].

Hence, the Grassmann manifolds $G_{n,2}$ are further in detail considered in [21], where it is proved that previously constructed model can be significantly simplified, that is, it is constructed simpler and much nicer model for the orbit space $G_{n,2}/T^n$. This is done by proving that the admissible polytopes, as well as the chamber decomposition they induce on $\Delta_{n,2}$, can be described using a suitable arrangement of hyperplanes in $\mathbb{R}^{n-1}$. The final result can be formulated as follows:

Theorem 18. *There exists continuous surjection $G : \Delta_{n,2} \times \mathcal{F}_n \to G_{n,2}/T^n$ such that $G_{n,2}/T^n$ is homeomorphic to the quotient space of $\Delta_{n,2} \times \mathcal{F}_n$ by the map G, where $\mathcal{F}_n$ is the universal space of parameters for the canonical T^n-action on $G_{n,2}$.*

Thus, in this case, the model space $U_n = \Delta_{n,2} \times \mathcal{F}_n$ for $G_{n,2}/T^n$ is, in a sense, a manifold with corners. In addition, the universal space of parameters $\mathcal{F}_n$ is explicitly described in [22] and this description purely comes from the equivariant topology of the Grassmannians $G_{n,2}$ using remarkable construction from algebraic geometry known as the wonderful compactification. We obtained that $\mathcal{F}_n$ is a smooth, compact manifold for which we proved to be diffeomorphic to the famous Deligne-Mumford compactification $\overline{\mathcal{M}}_{0,n}$ of the moduli space $\mathcal{M}_{0,n}$ of genus zero curves with n marked distinct points. The later one is proved in [36] to be diffeomorphic to the Chow quotient $G_{n,2}/\!/(\mathbb{C}^*)^n$ as defined in algebraic geometry.

In conclusion, the theory of $(2n, k)$-manifolds opens many new directions and provides the tools for their development. Some of these directions which are nowadays very current in equivariant topology and combinatorial topology would be adjustment of GKM theory to the set up of the theory of $(2n, k)$-manifolds or establishing connection between combinatorial subdivision of a convex polytope, which comes from a $(2n, k)$-manifold, and general polytope theory, in particular theory of hyperplane arrangements.

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TORUSNA DEJSTVA U TOPOLOGIJI

Sažetak

Izučavanje dejstva grupa je danas široko prisutno u svim oblastima matematike i to uglavnom sa dva aspekta. Proučavanje prostora orbita dejstva grupa sa stanovišta topologije, algebre, geometrije, kombinatorike se pokazuje važnim u raznim problemima matematike i matematičke fizike. S druge strane, prisustvo dejstva grupe na topološki, algebarski, geometrijski, kombinatorni objekat i izučavanje osobina tog dejstva u velikom broju slučajeva vodi ka značajnim rezultatima o samom objektu. U ovom radu dat je pregled nekih rezultata koji se odnose na dejstvo kompaktnog torusa na glatke mnogostrukosti. Posebna pažnja je posvećena rezultatima koji se odnose na kanonsko dejstvo kompaktnog torusa na kompaktne homogene prostore pozitivne Ojlerove karakteristike. U prvom redu to se odnosi na teoriju unitarnih kobordizama ovih prostora, kao i na konstrukciju kombinatorno-glatkih modela za njihov prostor orbita.