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ON THE FRICTIONAL IMPACT OF A SPINNING DISK AGAINST A FIXED SURFACE

Abstract

The expressions for the horizontal velocity component of the center of the disk and its angular velocity are derived for different incidence velocity conditions giving rise to either unidirectional slip, or the slip-stick transition during the impact process. It is shown that the works done by the normal and tangential reactive forces during the restitution phase are always smaller than the corresponding works during the compression phase of the impact. As a consequence, the frictional dissipation during the restitution phase is always smaller than during the compression phase. The lower and upper bounds on the work ratios are derived and shown to depend only on the coefficient of normal restitution. In the case when the slip-stick transition takes place during the impact, the tangential impact coefficient is shown to be dependent on the coefficient of normal restitution and the incidence velocities of the disk. A new appealing representation of the tangential impact coefficient in terms of the horizontal components of

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the incidence and rebound velocity of the contact point is then given. The kinetic energy is evaluated throughout the impact, demonstrating that its minimum is not necessarily attained at the end of the compression phase.

O FRIKCIONOM UDARU ROTIRAJUĆEG DISKA U NEPOKRETNU RAVAN

Sažetak

U radu su izvedeni izrazi za horizontalnu komponentu brzine centra diska i njegovu ugaonu brzinu tokom frikcionog udara diska u nepokretnu ravan za različite početne uslove, koji odgovaraju proklizavanju diska sa i bez kotrljanja. Pokazano je da su radovi normalne i tangencijalne sile udara za vrijeme restitucije uvijek manji od korespondentnih radova tokom kompresivne faze udara. Frikciona disipacija tokom restitucije je konsekventno uvijek manja od disipacije tokom kompresije. U slučaju kada je udar propraćen kombinovanim proklizavanjem i kotrljanjem diska, tangencijalni koeficijent udara zavisi od koeficijenta normalne restitucije i odnosa komponenti brzine kontaktne tačke udara diska u ravan. Promjena kinetičke energije tokom udara je izračunata, ilustrujući da se njen minimum ne dostiže uvijek na kraju kompresivne faze udara.

1. INTRODUCTION

The determination of the rebounding velocity components of colliding bodies is an old mechanics problem, with its origin in early work by Newton and Poisson. Newton defined the coefficient of the normal restitution as the ratio of the relative normal velocities after and before the impact,

$$\kappa_{\mathrm{N}} = \frac{(\mathbf{v}_{\mathrm{B}} - \mathbf{v}_{\mathrm{A}}) \cdot \mathbf{n}}{(\mathbf{v}_{\mathrm{A}}^0 - \mathbf{v}_{\mathrm{B}}^0) \cdot \mathbf{n}}\,, \tag{1.1}$$

where A and B label the two bodies colliding at the point over a tangent plane whose unit normal is $\mathbf{n}$. In contrast to Newton's kinematic definition, Poisson's kinetic definition is based on the ratio of the magnitudes of the normal impulses corresponding to the periods of restitution and compression,

$$\kappa_{\mathrm{P}} = \frac{\int_{t_0}^{t_1} N \, \mathrm{d}t}{\int_0^{t_0} N \, \mathrm{d}t} \, . \tag{1.2}$$

The total duration of the impact is t_1, and t_0 corresponds to the end of its compression phase, defined by the vanishing of the normal component of the relative velocity between the two bodies, $\mathbf{v}_{\mathrm{B}}(t_0) \cdot \mathbf{n} = \mathbf{v}_{\mathrm{A}}(t_0) \cdot \mathbf{n}$. The normal component of the reactive force between the colliding bodies is denoted by N.

In the absence of friction (frictionless impact), the Poisson definition of the coefficient of normal restitution yields the same expression, in terms of the relative velocities, as does the Newton definition ($\kappa_{\mathrm{P}} = \kappa_{\mathrm{N}}$), which is demonstrated in standard dynamics textbooks (*e.g.*, Kilmister and Reeve, 1966). In the presence of friction, however, the two definitions are, in general, not equivalent. The simplest theory of the frictional impact is that of Whittaker (1961), in which it is assumed that the frictional impulse is in the slip direction and is equal to the product of the coefficient of friction and the magnitude of the normal impulse. Kane (1984) observed that this theory leads to an increase of kinetic energy upon the impact of a double pendulum with a rough horizontal surface, for some values of the coefficients of friction and normal restitution, and for some kinematic parameters of motion. Keller (1986) explained this by noting that Whittaker's theory applies only when the direction of sliding is constant throughout the collision. If there is a reversal of the slip direction during the impact process, the coefficient of the proportionality between the tangential and normal impulse is different from by the coefficient of kinetic friction. Keller's analysis also demonstrated the advantage of using the normal impulse as an independent variable, instead of physical time, to cast and analyze the governing differential equations of motion during the impact process. Stronge (1990) introduced an energetic coefficient of normal restitution, whose square is equal to the negative ratio of the work done by the normal component of the impulsive reaction during the restitution and compression phases of the impact,

$$\eta^2 = -\frac{W_{\mathrm{r}}^n}{W_{\mathrm{c}}^n}\,. \tag{1.3}$$

Numerous papers, proposing different models of frictional impact, were published since, among which we refer to Wang and Mason (1992), Smith and Liu (1992), Ivanov (1992), Battle (1993), Brach (1997), Rubin (1998), Chatterjee and Ruina (1998), and Lankarani (2000). A comprehensive treatment of the subject, with a historical outline, can be found in the monographs or review articles by Brach (1991), Brogliato (1999), Stronge (2000), and Stewart (2000).

In the present paper we revisit the classical problem of the frictional impact of the spinning disk against a fixed surface. By employing Keller's method of analysis, we derive the expressions for the horizontal velocity component of the center of the disk and the angular velocity of the disk, in terms of the monotonically increasing normal impulse during the impact process. Different incidence velocity conditions give rise to either unidirectional slip, or the slip-stick transition during the impact process. In the studied problem, three different definitions of the coefficient of normal restitution (Newton's kinematic, Poisson's kinetic, and Stronge's energetic definition) are equivalent to each other. An analysis of the work done by the normal and tangential reactive forces reveal that the works during the restitution phase are always smaller than the works during the compression phase of the impact. As a consequence, the frictional dissipation during the restitution phase is always smaller than during the compression phase of the impact. The lower and upper bounds on the work ratios are derived and shown to depend only on the coefficient of normal restitution. The expression for the tangential impact coefficient, defined as the ratio of the tangential and normal component of the impulse, is then derived. In the case when the slip-stick transition takes place during the impact, the tangential impact coefficient is shown to be

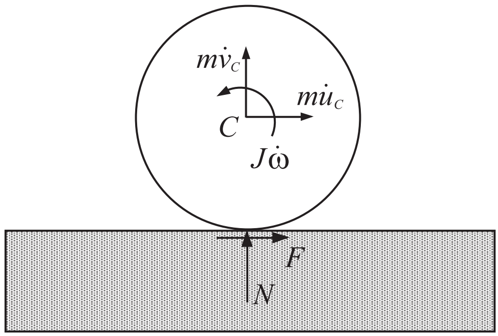

Figure 1: A spinning circular disk during an impact with a fixed horizontal surface. The reactive forces from the rough surface are N and F, the angular acceleration of the disk is $\dot{\omega}$, and the acceleration components of its mass center are $\dot{u}_C$ and $\dot{v}_C$.

dependent on the coefficient of normal restitution and the incidence velocities of the disk. Its upper and lower bound are equal to the positive and negative value of the kinetic coefficient of friction. A new appealing representation of the tangential impact coefficient in terms of the horizontal components of the incidence and rebound velocity of the contact point is then given. The kinetic energy is evaluated throughout the impact, demonstrating that its minimum is not necessarily attained at the end of the compression phase.

2. IMPACT ANALYSIS OF RIGID DISK STRIKING A FIXED SURFACE

Figure 1 shows a rigid circular disk (solid or hollow), or a sphere under plane motion, of outer radius R, mass m, and radius of gyration ρ, striking a fixed horizontal surface at small or moderate speeds, such that the extent of deformation around the contact point is localized and infinitesimally small. The angular velocity of the disk just before the impact is ω_-, and the incidence velocities of the mass center are u_C^- and v_C^-. If ω^+, u_C^+, and v_C^+ are the corresponding rebounding velocities, immediately after the impact of duration t_1, by the impulse principle we can write

$$mu_C^- + \int_0^{t_1} F\,\mathrm{d}t = mu_C^+\,, \tag{2.1}$$

$$mv_C^- + \int_0^{t_1} N\,\mathrm{d}t = mv_C^+\,, \tag{2.2}$$

$$J\omega^- + R\int_0^{t_1} F\,\mathrm{d}t = J\omega^+\,, \tag{2.3}$$

where $J = m\rho^2$ is the disk's moment of inertia about the center point C, and N and F are the normal and friction force acting on the disk at the contact point with the rough horizontal surface. By comparing (2.1) and (2.3), there is a connection $\zeta(u_C^+ - u_C^-) = R(\omega^+ - \omega^-)$, where $\zeta = 1 + R^2/\rho^2$.

The coordinates of the contact point change only infinitesimally during the time of the impact $t \in [0, t_1]$, so that the equations of motion during the impact are

$$m\,\frac{\mathrm{d}u_C}{\mathrm{d}t} = F\,, \quad m\,\frac{\mathrm{d}v_C}{\mathrm{d}t} = N\,, \quad J\,\frac{\mathrm{d}\omega}{\mathrm{d}t} = FR\,, \tag{2.4}$$

where u_C, v_C, and ω are the velocity components during the impact. The weight of the disk mg, as a nonimpulsive force, does not contribute to impulse equations. The horizontal velocity component of the contact point is related to the horizontal component of the velocity of the center of the disk by $u = u_C + R\omega$, and since $v = v_C$, equations (2.4) can be rewritten as[†]

$$m\left(\frac{\mathrm{d}u}{\mathrm{d}t} - R\,\frac{\mathrm{d}\omega}{\mathrm{d}t}\right) = F\,, \quad m\,\frac{\mathrm{d}v}{\mathrm{d}t} = N\,, \quad J\,\frac{\mathrm{d}\omega}{\mathrm{d}t} = FR\,. \tag{2.5}$$

[†]The tangential compliance of the contact region, which would bring an additional contribution to u, in addition to $u_C + R\omega$, is ignored. The assumption of zero tangential compliance (or infinite tangential stiffness) is usually a satisfactory assumption for the analysis of the rebound problems with large initial slip. At small slip, Maw *et al.* (1976) found, by using the Hertz contact theory for oblique impact between rough elastic spheres, that the contact area had an outer sliding annulus, while inner area had no relative tangential displacement (sticking area). They also showed that the direction of slip could be reversed during collision, which is the effect due to tangential compliance.

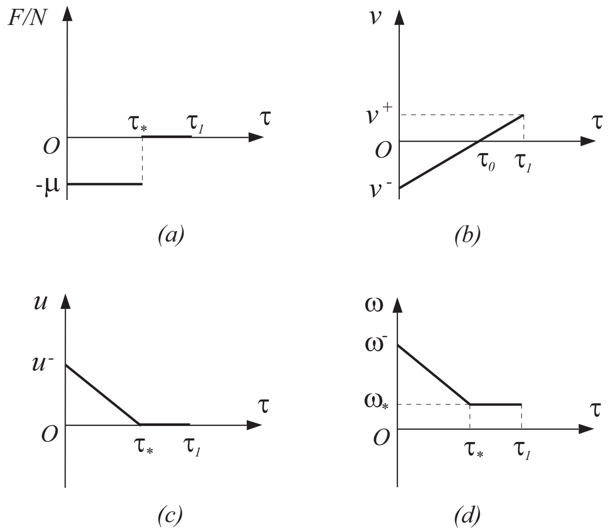

Figure 2: The variation of the force ratio (F/N), the velocity components of the contact point (u and v), and the angular velocity (ω) during the impact, in the case $\tau_* < \tau_1$, where τ_* is defined by $u(\tau_*) = 0$.

By comparing the first and third of these equations if follows that $\mathrm{d}u/\mathrm{d}t$ must be different from zero during the impact process; otherwise the equations are mutually incompatible, unless $F = 0$. Therefore, the impact process may include two stages of motion: a sliding stage ($\mathrm{d}u/\mathrm{d}t \neq 0$), during which the friction force is governed by Amontons–Coulomb's law of dry friction $F = -\mu N \,\mathrm{sign}(u)$ (Fig. 2a), where μ is the coefficient of kinetic friction, and a rolling (sticking) stage ($\mathrm{d}u/\mathrm{d}t = 0$), during which $F = 0$ (and thus $\omega =$ const. and $u_C = -R\omega =$ const.); Stronge (2000), pp. 55–57. The coefficient μ accounts for the roughness of the colliding bodies.

Following Keller (1986), the unknown time variation of the normal force $N(t)$ can be eliminated from the analysis by introducing a monotonically increasing impulse parameter

$$\tau = \int_0^t N \mathrm{d}t \,, \quad \mathrm{d}\tau = N \mathrm{d}t \,, \tag{2.6}$$

so that the equations (2.5) can be recast as

$$m(\mathrm{d}u - R\mathrm{d}\omega) = \frac{F}{N}\,\mathrm{d}\tau\,, \quad m\,\mathrm{d}v = \mathrm{d}\tau\,, \quad J\,\mathrm{d}\omega = R\,\frac{F}{N}\,\mathrm{d}\tau\,. \tag{2.7}$$

3. VERTICAL COMPONENT OF VELOCITY AND COEFFICIENT OF NORMAL RESTITUTION

By integrating the second equation in (2.7), the normal component of the velocity is

$$v = v^- + \frac{\tau}{m}\,, \quad 0 \leq \tau \leq \tau_1\,. \tag{3.1}$$

Let (t_0, τ_0) correspond to the end of the compression phase of the impact, defined by the condition $v(t_0) = 0$. From (3.1), this implies that

$$v^- = -\frac{\tau_0}{m}\,, \quad v^+ = \frac{\tau_1 - \tau_0}{m}\,. \tag{3.2}$$

The normal velocity component can thus be written as

$$v = v^-\left(1 - \frac{\tau}{\tau_0}\right), \quad 0 \leq \tau \leq \tau_1\,. \tag{3.3}$$

The following interpretation of the expressions (3.2) for the velocities v^- and v^+, based on the energy considerations, is helpful. From Fig. 2b, the compressive energy associated with the normal impulse is $-\tau_0 v^-/2$ (the area of the triangle under the τ-axis). This must be equal to the incidence kinetic energy $m(v^-)^2/2$; thus, $v^- = -\tau_0/m$. Similarly, the restitution energy $(\tau_1 - \tau_0)v^+/2$ (the area of the triangle above the τ-axis), being responsible for the liftoff of the disk, must be equal to $m(v^+)^2/2$; thus $v^+ = (\tau_1 - \tau_0)/m$.

By dividing the two expressions in (3.2), one has

$$-\frac{v^+}{v^-} = \frac{\tau_1 - \tau_0}{\tau_0}\,. \tag{3.4}$$

This shows that for the frictional (or nonfrictional) impact of the spinning disk against a fixed surface, the Newton kinematic definition of the coefficient of normal restitution ($\kappa_\mathrm{N} = -v^+/v^-$) and the Poisson kinetic definition ($\kappa_\mathrm{P} = \tau_1/\tau_0 - 1$) are equivalent to each other ($\kappa_\mathrm{N} = \kappa_\mathrm{P}$). The square of the Stronge energetic coefficient of normal restitution is defined as the negative ratio of the work done by the normal component of impulsive reactions during the restitution and compression phases of the impact,

$$\eta^2 = \frac{(\tau_1 - \tau_0)v^+/2}{-\tau_0 v^-/2} = -\frac{\tau_1 - \tau_0}{\tau_0}\,\frac{v^+}{v^-}\,. \tag{3.5}$$

Thus, in view of (3.4), all three definitions of the coefficient of normal restitution for a spinning disk striking a rough horizontal surface are equivalent,[‡] $\kappa_\mathrm{N} = \kappa_\mathrm{P} = \eta$. This common coefficient of normal restitution will be denoted in the sequel by κ. The corresponding terminal normal impulse is, from (2.2),

$$\tau_1 = (1+\kappa)\tau_0 = -(1+\kappa)mv^-\,. \tag{3.6}$$

The coefficient of normal restitution is dependent on v^-. For sufficiently small $|v^-|$, the energy associated with the vertical displacement during the impact may all be elastic and recoverable; for larger $|v^-|$ and the correspondingly larger normal force, inelastic compression takes place, which results in the smaller liftoff force during restitution, and thus smaller value of the coefficient κ. Since κ is assumed to be independent of μ, the functional dependence $\kappa = \kappa(v^-)$ can be determined from the sequence of experiments with vertically falling disks at different incidence velocities and zero spin. Having $\kappa(v^-)$ so determined, the terminal impulse $\tau_1(v^-) = -[1+\kappa(v^-)]mv^-$ applies for spinning disks striking the horizontal surface at any angle, under the same vertical velocity component v^-.

The upper bound on the coefficient of normal restitution is equal to one, *i.e.*, $\kappa \leq 1$, because the restitution phase of the impact cannot

[‡]For other impact problems, this is in general not the case. For example, for a rigid pendulum striking a rough surface, the energetic coefficient is a geometric mean of Newton's and Poisson's coefficients of normal restitution (Lubarda, 2010).

deliver more energy than what was stored during the compression phase. Thus,

$$\int_0^{\tau_1} v\,\mathrm{d}\tau = \frac{1}{2}\,\tau_1(v^- + v^+) \le 0\,, \tag{3.7}$$

which implies $-v^+/v^- \le 1$ and, therefore, the inequality $\kappa \le 1$. The limiting case $\kappa = 1$ corresponds to purely elastic compression; the dissipation of energy in this case is associated with the frictional sliding only, which affects u^+ and ω^+, but not $v^+ = -v^-$.

4. HORIZONTAL COMPONENT OF VELOCITY

Let (t_*, τ_*) correspond to the instant when the horizontal component of the velocity of the contact point vanishes $u(\tau_*) = 0$. In the sliding stage of the impact $[0, \tau_*]$, where $\tau_* < \tau_1$, by combining the first and third equation in (2.7), we have

$$m\,\mathrm{d}u = \zeta\,\frac{F}{N}\,\mathrm{d}\tau\,. \tag{4.1}$$

In the interval $\tau \in (0, \tau_*)$, u is of the same sing as u^-, so that $F/N = -\mu\,\mathrm{sign}(u^-)$, and the integration of (4.1) yields

$$u = u^- - \zeta\,\frac{\mu\tau}{m}\,\mathrm{sign}(u^-)\,, \quad 0 \le \tau \le \tau_*\,. \tag{4.2}$$

Imposing the condition $u(\tau_*) = 0$, the above specifies

$$\tau_* = \frac{mu^-\mathrm{sign}(u^-)}{\zeta\mu} = \frac{m|u^-|}{\zeta\mu}\,, \quad \frac{\tau_*}{\tau_0} = \frac{|u^-/v^-|}{\zeta\mu}\,, \tag{4.3}$$

where $|u^-| = u^-\,\mathrm{sign}(u^-)$, and

$$u^- = u_C^- + R\omega^-\,, \quad v^- = v_C^-\,. \tag{4.4}$$

Thus, (4.2) can be rewritten (Fig. 2c) as

$$u = u^-\left(1 - \frac{\tau}{\tau_*}\right), \quad 0 \le \tau \le \tau_*\,. \tag{4.5}$$

Note that

$$\tau_* \le \tau_1 \quad \Leftrightarrow \quad |u^-/v^-| \le (1+\kappa)\zeta\mu\,. \tag{4.6}$$

Similarly, by integration of the third equation in (2.7), there follows

$$\omega = \omega^- - (\zeta - 1)\,\frac{\mu\tau}{mR}\,\mathrm{sign}(u^-)\,, \quad 0 \le \tau \le \tau_*\,. \tag{4.7}$$

The expression for $\omega_* = \omega(\tau_*)$ follows by substituting (4.3) into (4.7), with the result

$$\omega_* = \omega^- - \left(1 - \zeta^{-1}\right)\frac{u^-}{R} = \frac{1}{R}\left(\zeta^{-1}\,u^- - u_C^-\right). \tag{4.8}$$

Thus, (4.7) can be rewritten (Fig. 2d) as

$$\omega = \omega^- - (\omega^- - \omega_*)\,\frac{\tau}{\tau_*}\,, \quad 0 \le \tau \le \tau_*\,. \tag{4.9}$$

The normal component of the velocity $v_* = v(\tau_*)$ is

$$v_* = v^-\left(1 - \frac{\tau_*}{\tau_0}\right) = v^- + \frac{|u^-|}{\zeta\mu}\,. \tag{4.10}$$

Finally, the horizontal component of the velocity of the center of the disk is obtained from $u_C = u - R\omega$ as

$$u_C = u_C^- - \frac{\mu\tau}{m}\,\mathrm{sign}(u^-) = u_C^- - (u_C^- + R\omega_*)\,\frac{\tau}{\tau_*}\,, \quad 0 \le \tau \le \tau_*\,. \tag{4.11}$$

In the rolling interval $\tau \in (\tau_*, \tau_1)$, $u = 0$, $F = 0$, and

$$\omega = \omega_*\,, \quad u_C = -R\omega_*\,, \quad \tau_* \le \tau \le \tau_1\,. \tag{4.12}$$

Thus, the rebounding velocity components are also $\omega^+ = \omega_*$ and

$$u_C^+ = -R\omega_* = u_C^- - \zeta^{-1}u^-\,, \quad |u^-/v^-| \le (1+\kappa)\zeta\mu\,. \tag{4.13}$$

The expression for u_C^+ is independent of the coefficient of friction and the coefficient of normal restitution, but the right-hand side of the inequality in (4.13) depends on μ and κ.

It is of interest is to determine the condition on the incidence velocity for which the compression phase of the impact ends before the sliding changes into the rolling ($\tau_0 \leq \tau_*$). By using $\tau_0 = -mv_C^-$ and (4.3), this condition is

$$|u^-/v^-| \geq \zeta\mu\,, \quad |v_C^-| \leq \frac{|u_C^- + R\omega^-|}{\zeta\mu}\,. \tag{4.14}$$

The sliding prevails throughout the impact if $\tau_* \geq \tau_1$, *i.e.*,

$$|u^-/v^-| \geq (1+\kappa)\zeta\mu\,, \quad |v_C^-| \leq \frac{|u_C^- + R\omega^-|}{(1+\kappa)\zeta\mu}\,. \tag{4.15}$$

In this case, (4.5) and (4.9) hold in the entire impact interval $[0, \tau_1]$, and the rebounding velocities are

$$\begin{aligned} u^+ &= u^-\left(1 - \frac{\tau_1}{\tau_*}\right), \\ \omega^+ &= \omega^- - \left(1 - \zeta^{-1}\right)\frac{u^-}{R}\,\frac{\tau_1}{\tau_*}\,, \\ u_C^+ &= u_C^- - \zeta^{-1}u^-\,\frac{\tau_1}{\tau_*}\,, \end{aligned} \tag{4.16}$$

where

$$\frac{\tau_1}{\tau_*} = \frac{(1+\kappa)\zeta\mu}{|u^-/v^-|} \leq 1\,. \tag{4.17}$$

In summary, the ratio of the horizontal velocity components is

$$\frac{u_C^+}{u_C^-} = \begin{cases} 1 - \zeta^{-1}\,\dfrac{u^-}{u_C^-}, & |u^-/v^-| \leq (1+\kappa)\zeta\mu\,, \\[2ex] 1 - \dfrac{(1+\kappa)\mu}{|u^-/v^-|}\,\dfrac{u^-}{u_C^-}, & |u^-/v^-| \geq (1+\kappa)\zeta\mu\,. \end{cases}$$

The ratio of the rebound and the incidence angular velocity is

$$\frac{\omega^+}{\omega^-} = \begin{cases} 1 - \left(1 - \zeta^{-1}\right)\dfrac{u^-}{R\omega^-}, & |u^-/v^-| \leq (1+\kappa)\zeta\mu\,, \\[2ex] 1 - (\zeta - 1)\,\dfrac{(1+\kappa)\mu}{|u^-/v^-|}\,\dfrac{u^-}{R\omega^-}, & |u^-/v^-| \geq (1+\kappa)\zeta\mu\,. \end{cases}$$

The determination of the conditions between the incidence velocities in order that either a backward rebound, or a rebound with the reversed spin, or a spinning or nonspinning vertical rebound, takes place is straightforward and is here omitted for brevity.

5. ENERGY DISSIPATION DURING THE IMPACT

The energy dissipated during the impact process is equal to the negative work done by the reactive forces N and F on the corresponding displacements during the impact,

$$\Delta E = -\int_0^{t_1} N v \, \mathrm{d}t - \int_0^{t_1} F u \, \mathrm{d}t = -\int_0^{\tau_1} v \, \mathrm{d}\tau - \int_0^{\tau_1} \frac{F}{N} u \, \mathrm{d}\tau \,. \quad (5.1)$$

In the case when the impact involves a slip followed by stick ($\tau_* \leq \tau_1$), the above is

$$\Delta E = -\int_0^{\tau_1} v \, \mathrm{d}\tau + \mu \, \mathrm{sign}(u^-) \int_0^{\tau_*} u \, \mathrm{d}\tau \,. \quad (5.2)$$

From Figures 2(b) and (c),

$$\int_0^{\tau_1} v \, \mathrm{d}\tau = \frac{1}{2}(v^- + v^+)\tau_1 \,, \quad \int_0^{\tau_*} u \, \mathrm{d}\tau = \frac{1}{2}\tau_* u^- \,. \quad (5.3)$$

By using the expression (4.3) for τ_*, the substitution of (5.3) into (5.2) yields

$$\Delta E = \frac{1}{2}\tau_1 |v^-| \left(2 - \frac{\tau_1}{\tau_0} + \zeta^{-1} \frac{\tau_0}{\tau_1} |u^-/v^-|^2 \right), \quad \tau_* \leq \tau_1 \,, \quad (5.4)$$

i.e,

$$\Delta E = \frac{1}{2} m |v^-|^2 \left(1 - \kappa^2 + \zeta^{-1} |u^-/v^-|^2\right), \quad |u^-/v^-| \leq (1+\kappa)\zeta\mu \,.$$

This expression for the energy loss is independent of μ. Higher the coefficient of friction, higher the friction force, but smaller the impulse τ_* at which $u = 0$, with the net outcome that $\mu\tau_*$, contributing to (5.2) via (5.3), is independent of μ. Thus, the μ-dependence of ΔE is all

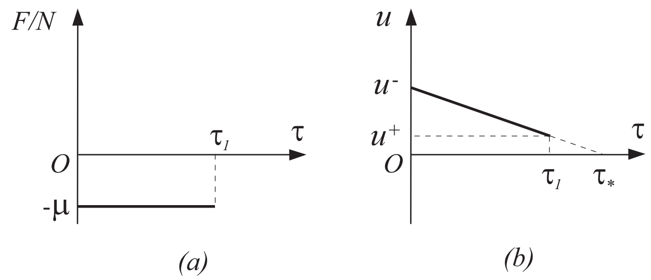

Figure 3: The variation of the force ratio (F/N) and the horizontal component of the velocity of the contact point (u) during the impact, in the case $\tau_* > \tau_1$.

embedded in the μ-dependence of the right-hand side of the inequality $|u^-/v^-| \leq (1+\kappa)\zeta\mu$.

If the sliding prevails throughout the impact $(\tau_* \geq \tau_1)$, Fig. 3, then

$$\int_0^{\tau_1} \frac{F}{N}\, u \,\mathrm{d}\tau = -\mu \,\mathrm{sign}(u^-) \int_0^{\tau_1} u \,\mathrm{d}\tau\,, \quad \int_0^{\tau_1} u \,\mathrm{d}\tau = \frac{1}{2}\,(u^- + u^+)\tau_1\,.$$

The dissipated energy in this case is

$$\Delta E = \frac{1}{2}\,\tau_1 |v^-| \left(2 - \frac{\tau_1}{\tau_0}\right) + \frac{1}{2}\,\mu\tau_1 |u^-| \left(2 - \frac{\tau_1}{\tau_*}\right), \quad \tau_* \geq \tau_1\,, \tag{5.5}$$

or

$$\Delta E = \frac{1}{2}\, m|v^-|^2 (1+\kappa) \left\{1 - \kappa + \mu \left[2\,|u^-/v^-| - (1+\kappa)\zeta\mu\right]\right\}, \tag{5.6}$$

which applies for $|u^-/v^-| \geq (1+\kappa)\zeta\mu$.

An alternative derivation of the expression for the dissipated energy is as follows. From Section 5, the total horizontal (frictional) impulse can be written as

$$f_1 = \int_0^{t_1} F \,\mathrm{d}t = -\mu \,\mathrm{sign}(u^-) \min(\tau_*, \tau_1)\,. \tag{5.7}$$

The dissipated work of the frictional force on the horizontal displacement is

$$W^t = \int_0^{t_1} F u \,\mathrm{d}t = \frac{1}{2}\, f_1 (u^- + u^+)\,, \tag{5.8}$$

where $u^+ = 0$ if $\tau_* \leq \tau_1$. Since the work of the normal force on the vertical displacement is

$$W^n = \int_0^{\tau_1} v \, \mathrm{d}\tau = \frac{1}{2}\,\tau_1(v^- + v^+)\,, \tag{5.9}$$

the entire dissipated energy, $\Delta E = -(W^t + W^n)$, can be expressed as

$$\Delta E = -\frac{1}{2}\,f_1(u^- + u^+) - \frac{1}{2}\,\tau_1(v^- + v^+)\,. \tag{5.10}$$

If $f_1 = 0$, this reduces to the Thomson and Tait formula for the frictionless impact (Brogliato, 1996).

Yet another derivation of (5.10) is deduced by expressing the dissipated energy as the difference of the incidence and rebounding kinetic energies. This can be conveniently written (Stronge, 2000) as

$$\begin{aligned}\Delta E = \frac{1}{2}\,J(\omega^- - \omega^+)(\omega^- + \omega^+) + \frac{1}{2}\,m(u_C^- - u_C^+)(u_C^- + u_C^+) \\ + \frac{1}{2}\,m(v_C^- - v_C^+)(v_C^- + v_C^+)\,.\end{aligned} \tag{5.11}$$

By the impulse equations (2.1)–(2.3), we have

$$J(\omega^- - \omega^+) = -Rf_1\,, \quad m(u_C^- - u_C^+) = -f_1\,, \quad m(v_C^- - v_C^+) = -\tau_1\,,$$

and since $u = u_C + R\omega$ and $v = v_C$, the substitution into (5.11) reproduces (5.10).

6. COMPRESSION AND RESTITUTION WORKS

The works done by the impulsive reactions during compression and restitution phases of the impact are

$$W_{\mathrm{c}} = \int_0^{t_0} Nv \, \mathrm{d}t + \int_0^{t_0} Fu \, \mathrm{d}t = \int_0^{\tau_0} v \, \mathrm{d}\tau + \int_0^{\tau_0} \frac{F}{N}\,u \, \mathrm{d}\tau\,, \tag{6.1}$$

$$W_{\mathrm{r}} = \int_{t_0}^{t_1} Nv \, \mathrm{d}t + \int_{t_0}^{t_1} Fu \, \mathrm{d}t = \int_{\tau_0}^{\tau_1} v \, \mathrm{d}\tau + \int_{\tau_0}^{\tau_1} \frac{F}{N}\,u \, \mathrm{d}\tau\,. \tag{6.2}$$

Here, the end of the compression phase is defined by condition $v(\tau_0) = 0$. The total work is $W = W_\mathrm{c} + W_\mathrm{r}$, such that $E_- + W = E_+$. It is recalled from (3.3) and (4.5) that $v = v^-(1 - \tau/\tau_0)$ for all τ, while

$$u = \begin{cases} u^-(1 - \tau/\tau_*)\,, & 0 \le \tau \le \tau_*\,, \\ 0\,, & \tau_* \le \tau \le \tau_1\,, \end{cases} \tag{6.3}$$

and

$$\frac{F}{N} = \begin{cases} -\mu\,\mathrm{sign}(u^-)\,, & 0 \le \tau \le \tau_*, \\ 0\,, & \tau_* \le \tau \le \tau_1. \end{cases} \tag{6.4}$$

If $\tau_* \ge \tau_1$, then $F/N = -\mu\,\mathrm{sign}(u^-)$ in the entire interval $[0, \tau_1]$.

Independently of τ_*, the v-integrals are

$$W_\mathrm{c}^n = \int_0^{\tau_0} v\,\mathrm{d}\tau = \frac{1}{2}\,\tau_0 v^-\,, \quad W_\mathrm{r}^n = \int_{\tau_0}^{\tau_1} v\,\mathrm{d}\tau = -\frac{1}{2}\,\tau_0 v^- \left(\frac{\tau_1}{\tau_0} - 1\right)^2.$$

These two work contributions appear in Stronge's definition of the energetic coefficient of normal restitution,

$$\eta^2 = -\frac{W_\mathrm{r}^n}{W_\mathrm{c}^n} = \left(\frac{\tau_1}{\tau_0} - 1\right)^2 = \kappa^2\,, \tag{6.5}$$

demonstrating that for the spinning disk striking a rough fixed surface, $\eta = \kappa$ (Poisson's or Newton's coefficient of normal restitution). As discussed earlier, the restitution phase of the impact cannot deliver more energy than what was stored during the compression phase, $-W_\mathrm{r}^n/W_\mathrm{c}^n \le 1$, so that $\kappa \le 1$.

Since $m(v^-)^2/2 + W_\mathrm{c}^n + W_\mathrm{r}^n = m(v^+)^2/2$, the dissipated energy associated with the vertical motion of the disk is $\Delta E^n = -(1-\eta^2)W_\mathrm{c}^n$, so that there is an alternative but equivalent definition of the energetic coefficient of restitution, $\eta^2 = 1 + \Delta E^n/W_\mathrm{c}^n$.

The analysis of the work on the u displacement includes two cases. *Case I* ($\tau_* \le \tau_1$). This case involves two distinct subcases: $\tau_* \le \tau_0$ and $\tau_* \ge \tau_0$. If $\tau_* \le \tau_0$, the compression and restitution works are

$$W_\mathrm{c} = \frac{1}{2}\,\tau_0 v^- - \frac{1}{2}\,\mu\tau_*|u^-|\,, \quad W_\mathrm{r} = -\frac{1}{2}\,\tau_0 v^- \left(1 - \frac{\tau_1}{\tau_0}\right)^2. \tag{6.6}$$

Upon the substitution of the expressions for τ_* and τ_1, this becomes

$$W_{\rm c} = -\frac{1}{2}\,\tau_0|v^-|\left(1+\zeta^{-1}\,|u^-/v^-|^2\right), \quad W_{\rm r} = \frac{1}{2}\,\tau_0|v^-|\kappa^2\,, \tag{6.7}$$

which applies for $|u^-/v^-| \leq \zeta\mu$. If $\tau_* \geq \tau_0$, the works are

$$\begin{aligned} W_{\rm c} &= \frac{1}{2}\,\tau_0 v^- - \frac{1}{2}\,\mu\tau_0|u^-|\left(2-\frac{\tau_0}{\tau_*}\right), \\ W_{\rm r} &= -\frac{1}{2}\,\tau_0 v^-\left(1-\frac{\tau_1}{\tau_0}\right)^2 - \frac{1}{2}\,\mu\tau_*|u^-|\left(1-\frac{\tau_0}{\tau_*}\right)^2. \end{aligned} \tag{6.8}$$

This can be simplified to

$$\begin{aligned} W_{\rm c} &= -\frac{1}{2}\,\tau_0|v^-|\left(1-\zeta\mu^2+2\mu\,|u^-/v^-|\right), \\ W_{\rm r} &= \frac{1}{2}\,\tau_0|v^-|\left[\kappa^2-\zeta\mu^2\left(\frac{1}{\zeta\mu}\,|u^-/v^-|-1\right)^2\right], \end{aligned} \tag{6.9}$$

which applies for $\zeta\mu \leq |u^-/v^-| \leq (1+\kappa)\zeta\mu$.
Case II ($\tau_* \geq \tau_1$). In this case, $F/N = -\mu\,\mathrm{sign}(u^-)$ for all $\tau \in [0,\tau_1]$, and $\tau_0 < \tau_*$. The compression and restitution works are

$$\begin{aligned} W_{\rm c} &= \frac{1}{2}\,\tau_0 v^- - \frac{1}{2}\,\mu\tau_0|u^-|\left(2-\frac{\tau_0}{\tau_*}\right), \\ W_{\rm r} &= -\frac{1}{2}\,\tau_0 v^-\left(1-\frac{\tau_1}{\tau_0}\right)^2 - \frac{1}{2}\,\mu(\tau_1-\tau_0)|u^-|\left(2-\frac{\tau_0+\tau_1}{\tau_*}\right). \end{aligned} \tag{6.10}$$

After eliminating τ_* and τ_1, this becomes

$$\begin{aligned} W_{\rm c} &= -\frac{1}{2}\,\tau_0|v^-|\left(1-\zeta\mu^2+2\mu\,|u^-/v^-|\right), \\ W_{\rm r} &= \frac{1}{2}\,\tau_0|v^-|\left[\kappa^2+\kappa\zeta\mu^2\left(2+\kappa-\frac{2}{\zeta\mu}\,|u^-/v^-|\right)\right], \end{aligned} \tag{6.11}$$

which applies for $|u^-/v^-| \geq (1+\kappa)\zeta\mu$.

6.1. An analysis of the restitution work

It is easily verified that the work done during the compression phase of the impact is always negative, $W_{\rm c} < 0$. The work done during the restitution phase, however, can be either positive or negative, depending on the incidence velocity ratio and the coefficient of friction. For $\kappa > 0$, the following results apply in different intervals of $|u^-/v^-|$. For $|u^-/v^-| \leq \zeta\mu$, the restitution work is positive,

$$W_\mathrm{r} > 0\,, \quad |u^-/v^-| \leq \zeta\mu\,. \tag{6.12}$$

In the interval $\zeta\mu \leq |u^-/v^-| \leq (1+\kappa)\zeta\mu$, the restitution work can be either positive or negative, according to

$$\begin{aligned} &W_\mathrm{r} \geq 0\,, \quad \zeta\mu \leq |u^-/v^-| \leq \zeta\mu + \sqrt{\zeta}\,\kappa\,, \\ &W_\mathrm{r} \leq 0\,, \quad \zeta\mu + \sqrt{\zeta}\,\kappa \leq |u^-/v^-| \leq (1+\kappa)\zeta\mu\,, \quad \zeta\mu^2 \geq 1\,. \end{aligned} \tag{6.13}$$

In the second expression above, the condition $\zeta\mu^2 \geq 1$ is imposed in order that the right-hand side of the preceding inequality is greater or equal to the left-hand side. In the remaining interval $|u^-/v^-| \geq (1+\kappa)\zeta\mu$, the sign of W_r is determined by the condition on $\zeta\mu^2$, such that

$$\begin{aligned} &W_\mathrm{r} \leq 0\,, \quad |u^-/v^-| \geq (1+\kappa)\zeta\mu\,, \quad \zeta\mu^2 \geq 1\,, \\ &W_\mathrm{r} \geq 0\,, \quad |u^-/v^-| \geq (1+\kappa)\zeta\mu\,, \quad \zeta\mu^2 \leq 1\,. \end{aligned} \tag{6.14}$$

The works by the normal reaction during the compression and restitution phases of the impact are related by $W_\mathrm{r}^n = -\kappa^2 W_\mathrm{c}^n$, which follows from the definition of the energetic coefficient of normal restitution. In particular, W_r^n is never greater than W_c^n. We next prove that the work done by the frictional reaction during the restitution phase is always smaller than the frictional work during the compression phase of the impact.

First, it readily follows that

$$\frac{W_\mathrm{r}^t}{W_\mathrm{c}^t} = 0\,, \quad |u^-/v^-| \leq \zeta\mu\,, \tag{6.15}$$

because $F = 0$ during the restitution with $\tau_* \leq \tau_0$. In the next interval of $|u^-/v^-|$, the work ratio is

$$\frac{W_\mathrm{r}^t}{W_\mathrm{c}^t} = \frac{1}{\zeta\mu}\,\frac{|u^-/v^-|^2}{2|u^-/v^-| - \zeta\mu} - 1\,, \quad \zeta\mu \leq |u^-/v^-| \leq 1+\kappa)\zeta\mu\,. \tag{6.16}$$

By the analysis of the right-hand side, the bounds on this work ratio are

$$0 \leq \frac{W_{\mathrm{r}}^{t}}{W_{\mathrm{c}}^{t}} \leq \frac{\kappa^2}{1+2\kappa}\,. \tag{6.17}$$

The lower bound is attained for $|u^-/v^-| = \zeta\mu$, and the upper bound for $|u^-/v^-| = (1+\kappa)\zeta\mu$. Since $\kappa \leq 1$, it clearly follows that $W_{\mathrm{r}}^{t} < W_{\mathrm{c}}^{t}$, *i.e.*, the frictional work during restitution is smaller than during compression. For example, if $|u^-/v^-| = (1+\kappa)\zeta\mu$, the ratio $W_{\mathrm{r}}^{t}/W_{\mathrm{c}}^{t}$ is equal to 1/3 if $\kappa = 1$; 1/8 if $\kappa = 1/2$; and 1/120 if $\kappa = 1/10$.

In the remaining interval of $|u^-/v^-|$, the work ratio is found to be

$$\frac{W_{\mathrm{r}}^{t}}{W_{\mathrm{c}}^{t}} = \kappa\left[1 - \frac{(1+\kappa)\zeta\mu}{2|u^-/v^-| - \zeta\mu}\right], \quad |u^-/v^-| \geq (1+\kappa)\zeta\mu\,, \tag{6.18}$$

which is bounded by

$$\frac{\kappa^2}{1+2\kappa} \leq \frac{W_{\mathrm{r}}^{t}}{W_{\mathrm{c}}^{t}} \leq \kappa\,. \tag{6.19}$$

Both lower and upper bound are dependent only on the coefficient of normal restitution κ, independently of the parameter $\zeta\mu$. For $\kappa < 1$, it follows that $W_{\mathrm{r}}^{t} < W_{\mathrm{c}}^{t}$. There is more frictional dissipation during the compression than the restitution phase of the impact. This conclusion is supported by an intuitive physical anticipation from the outset of the analysis.

7. TANGENTIAL IMPACT COEFFICIENT

The tangential impulse at an arbitrary stage of the impact process is

$$f(\tau) = \int_0^t F\,\mathrm{d}t = \int_0^\tau \frac{F}{N}\,\mathrm{d}\tau\,. \tag{7.1}$$

If $\tau_* \leq \tau_1$, the force ratio is $F/N = -\mu\,\mathrm{sign}(u^-)$ for $\tau < \tau_*$, and zero otherwise. If $\tau_* \geq \tau_1$, then $F/N = -\mu\,\mathrm{sign}(u^-)$ for all $0 \leq \tau \leq \tau_1$. Consequently,

$$f(\tau) = -\mu\,\mathrm{sign}(u^-)\begin{cases} \tau\,, & 0 \leq \tau \leq \tau_*\,, \\ \tau_*\,, & \tau_* \leq \tau \leq \tau_1\,, \end{cases} \tag{7.2}$$

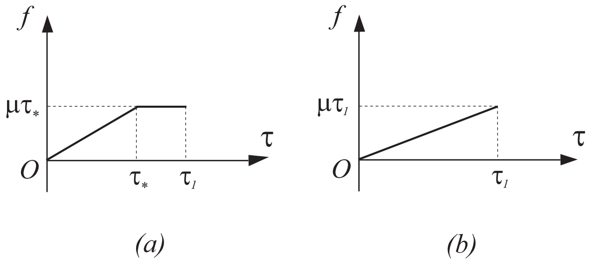

Figure 4: Routh's impact diagram showing the variation of the tangential impulse f *vs.* the normal impulse τ, in the case $u^- < 0$: (a) $\tau_* < \tau_1$; (b) $\tau_* > \tau_1$.

and

$$f(\tau) = -\mu\,\mathrm{sign}(u^-)\tau\,, \quad 0 \le \tau \le \tau_1 \le \tau_*\,. \tag{7.3}$$

The corresponding Routh's impact diagram is shown in Fig. 4.

Brach (1991) defines the tangential impact coefficient as the ratio of the tangential and normal component of the impulse,

$$\hat{\mu} = \frac{f_1}{\tau_1} = \frac{\int_0^{t_1} F\mathrm{d}t}{\int_0^{t_1} N\mathrm{d}t} = \frac{\int_0^{\tau_1} (F/N)\,\mathrm{d}\tau}{\tau_1}\,. \tag{7.4}$$

Thus, $\hat{\mu}$ can also be interpreted as the average (mean) value of the force ratio F/N over the impact interval τ_1. Another interpretation follows from (2.1) and (2.2), which show that $\hat{\mu}$ is the ratio of the horizontal and vertical velocity changes of the center of the disk due to the impact,

$$\hat{\mu} = \frac{f_1}{\tau_1} = \frac{u_C^+ - u_C^-}{v_C^+ - v_C^-}\,. \tag{7.5}$$

From (7.2) and (7.3), the terminal horizontal impulse is

$$f_1 = f(\tau_1) = -\mu\,\mathrm{sign}(u^-)\begin{cases} \tau_*\,, & \tau_* \le \tau_1\,, \\ \tau_1\,, & \tau_* \ge \tau_1\,, \end{cases} \tag{7.6}$$

and the substitution into (7.4) gives

$$\hat{\mu} = -\mu\,\mathrm{sign}(u^-)\begin{cases} \tau_*/\tau_1\,, & \tau_* \le \tau_1\,, \\ 1\,, & \tau_* \ge \tau_1\,. \end{cases} \tag{7.7}$$

Therefore, by using (4.3) for $\mu\tau_*$,

$$\hat{\mu} = \zeta^{-1}\,\frac{\tau_0}{\tau_1}\,\frac{u^-}{v^-} = \frac{1}{(1+\kappa)\zeta}\,\frac{u^-}{v^-}\,, \quad |u^-/v^-| \leq (1+\kappa)\zeta\mu\,. \tag{7.8}$$

The ratio of the magnitude of the tangential and normal component of the impulse is in this case not equal to μ (as in a simple Whittaker's (1961) and Kane and Levinson's (1985) theory of frictional impact), because there was a change from slip to stick motion during the frictional impact. On the other hand,

$$\hat{\mu} = -\mu\,\mathrm{sign}(u^-)\,, \quad |u^-/v^-| \geq (1+\kappa)\zeta\mu\,. \tag{7.9}$$

The magnitude of the ratio of the tangential and normal component of the impulse is in this case equal to μ, because there was a unidirectional slip during the entire frictional impact.

The value of $\hat{\mu}$ is clearly bounded by $-\mu \leq \hat{\mu} \leq \mu$. The bounds are attained when $|u^-/v^-| = (1+\kappa)\zeta\mu$; the lower bound if $\mathrm{sign}(u^-)$ is negative, and the upper bound if $\mathrm{sign}(u^-)$ is positive. The magnitude of the frictional force during the impact in these two cases is of equal magnitude (μN), but opposite direction, corresponding to unidirectional, forward or backward slip throughout the impact process.

Remarkably, both expressions (7.8) and (7.9) can be given a common representation in terms of the horizontal components of the incidence and rebound velocity of the contact point, which is

$$\hat{\mu} = -\mu\,\frac{|u^-|}{u^- - u^+}\,. \tag{7.10}$$

This can be easily verified by using $u^+ = u_C^+ + R\omega^+$, and the expressions for u_C^+ and ω^+ derived in Section 4, with the end result

$$u^+ = u^- \begin{cases} 1 - \dfrac{(1+\kappa)\zeta\mu}{|u^-/v^-|}\,, & |u^-/v^-| \geq (1+\kappa)\zeta\mu\,, \\[2ex] 0\,, & |u^-/v^-| \leq (1+\kappa)\zeta\mu\,. \end{cases} \tag{7.11}$$

8. DISCUSSION

There are other interesting aspects of the frictional impact of the spinning disk against a rough surface that take place during a short duration of the impact. For example, the end of the compression phase of the impact, defined by the condition $v(\tau_0) = 0$, does not have to be necessarily the state of the minimum kinetic energy, because of an interplay of the varying energy contributions from the horizontal and vertical component of the velocity of the center of the disk, and its angular velocity. This fact could be used to introduce other definition of the energetic coefficient of restitution, if needed to better match the experimental data (Ivanov, 1992). To determine the minimum kinetic energy during the impact, we use the work principle to express the kinetic energy $E(\tau)$ at an arbitrary instant of the impact process as the sum of the incidence kinetic energy E^- and the work done by the impulsive reactions,

$$E(\tau) = E^- + \int_0^\tau (v + uF/N)\,\mathrm{d}\tau\,. \tag{8.1}$$

Upon using the velocity expression $v = v^-(1 - \tau/\tau_0)$ and (6.3) for u, the integration in (8.1) gives

$$\frac{E - E^-}{\frac{1}{2}\,m\,|v^-|^2} = -\frac{\tau}{\tau_0}\left[2\left(1 + \mu\,|u^-/v^-|\right) - (1 + \zeta\mu^2)\,\frac{\tau}{\tau_0}\right], \quad 0 \leq \frac{\tau}{\tau_0} \leq \frac{\tau_*}{\tau_0}\,,$$

$$\frac{E - E^-}{\frac{1}{2}\,m\,|v^-|^2} = -\left[\frac{\tau}{\tau_0}\left(2 - \frac{\tau}{\tau_0}\right) + \zeta^{-1}\,|u^-/v^-|^2\right], \quad \frac{\tau_*}{\tau_0} \leq \frac{\tau}{\tau_0} \leq \frac{\tau_1}{\tau_0}\,.$$

The kinetic energy is at minimum at $\tau = \tau_{\mathrm{m}}$, when $v + Fu/N = 0$. Suppose that $\tau_{\mathrm{m}} \leq \tau_* \leq \tau_1$, so that $F/N = -\mu\,\mathrm{sign}(u^-)$, while v and u are given by (3.1) and (4.2). Then,

$$\tau_{\mathrm{m}} = m\,\frac{|v^-| + \mu|u^-|}{1 + \zeta\mu^2}\,. \tag{8.2}$$

This can be rewritten as either of

$$\frac{\tau_{\mathrm{m}}}{\tau_0} = \frac{1 + \mu\,|u^-/v^-|}{1 + \zeta\mu^2}\,, \quad \frac{\tau_{\mathrm{m}}}{\tau_*} = \frac{\zeta\mu}{1 + \zeta\mu^2}\left(\mu + \frac{1}{|u^-/v^-|}\right), \tag{8.3}$$

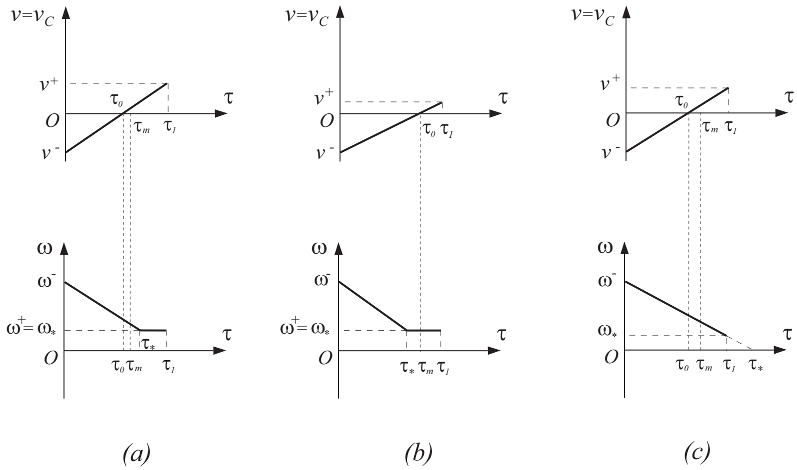

Figure 5: The kinetic energy attains its minimum during (a) the sliding phase of the impact if $|u^-/v^-| > \zeta\mu$, and in that case $\tau_0 < \tau_{\rm m} < \tau_* < \tau_1$; (b) the rolling phase of the impact if $|u^-/v^-| < \zeta\mu$, and in that case $\tau_* < \tau_{\rm m} = \tau_0 < \tau_1$; (c) after the end of the compression phase, in the case of sliding throughout the impact.

where $\tau_0 = m|v^-|$ and $\tau_* = m|u^-|/(\zeta\mu)$, with $\mu \neq 0$. To make the second expression in (8.3) compatible with $\tau_{\rm m} \leq \tau_*$, we must have $|u^-/v^-| \geq \zeta\mu$. This implies, from the first of (8.3), that $\tau_{\rm m} > \tau_0$ (Fig. 5a). Therefore, in this case $\tau_{\rm m}$ is in the range $\tau_0 \leq \tau_{\rm m} \leq \tau_*$. The corresponding minimum kinetic energy is

$$E_{\rm m} = E^- - \frac{1}{2}\, m|v^-|^2\, \frac{(1+\mu\,|u^-/v^-|)^2}{1+\zeta\mu^2}\,, \quad |u^-/v^-| \geq \zeta\mu\,. \tag{8.4}$$

If $\tau_{\rm m} < \tau_*$, but $\tau_* > \tau_1$ (so that slip prevails throughout the impact), we have $\tau_m < \tau_1$. Together, this implies that $\tau_{\rm m}$ is given by (8.3), provided that

$$(1+\kappa)\zeta\mu < |u^-/v^-| < \mu^{-1}\left[(1+\kappa)(1+\zeta\mu^2) - 1\right]. \tag{8.5}$$

In this case, from the first of (8.3), it follows that $\tau_{\rm m} > \tau_0$ (Fig. 5c). Thus, in this case the kinetic energy attains its minimum after the end of the compression phase.

Finally, if $\tau_1 > \tau_{\rm m} \geq \tau_*$, the kinetic energy attains its minimum during the rolling (sticking) portion of the impact ($u = 0$, $F = 0$), and $\tau_{\rm m} = -mv^- = \tau_0$ (Fig 5b). Since we have assumed that $\tau_{\rm m} \geq \tau_*$, in this case $|u^-/v^-| \leq \zeta\mu$, and the kinetic energy is minimum at the end of the compression phase of the impact, being equal to

$$E_{\rm m} = E^- - \frac{1}{2}\, m|v^-|^2 \left(1 + \zeta^{-1}\,|u^-/v^-|^2\right), \quad |u^-/v^-| \leq \zeta\mu\,. \tag{8.6}$$

We end this discussion by noting that the rebound analysis of a material particle striking a rough surface is dynamically indeterminate, and that a naive application of Whittaker's theory, based on the assumption that unidirectional slip takes place throughout the impact, can result in an unrealistic increase of kinetic energy. Indeed, the impulse equations for the particle striking a rough surface are $mu^- + f_1 = mu^+$ and $mv^- + \tau_1 = mv^+$ where τ_1 is the total impulse in the normal direction, and f_1 is the total frictional impulse during the impact. Here, $v^- < 0$ and, without loss of generality, we can assume that $u^- > 0$, so that the particle is hitting the surface coming from the left. There are four unknown quantities in the above two equations, two rebound velocity components and two impulse components. An additional equation is generated by introducing the coefficient of normal restitution κ, such that $v^+ = -\kappa v^-$ and $\tau_1 = -m(1+\kappa)v^-$. The frictional impulse f_1, and the horizontal component of the rebound velocity u^+ remain unknown. Without knowing kinetic (slip-stick) details of the frictional process during the impact, and without introducing additional impact coefficient relating u^+ and u^-, the problem is dynamically indeterminate. If one attempts to unfold this indeterminacy by assuming that a continuous unidirectional slip takes place throughout the impact process, so that f_1 is related to τ_1 by $f_1 = -\mu\tau_1$, the horizontal component of the rebound velocity becomes $u^+ = u^- + (1+\kappa)\mu v^-$. For some incidence velocities, this expression can be satisfactory, but for others it can lead to an unrealistic energy increase by the impact process. Indeed, the change of kinetic energy by frictional impact is

$$E^{+} - E^{-} = \frac{1}{2}\, m(v^{-})^{2}(1+\kappa)\left[\kappa - 1 + (1+\kappa)\mu^{2} - 2\mu\, \frac{u^{-}}{|v^{-}|}\right]. \quad (8.7)$$

This must be non-positive, which is the case if the tangent of the incidence angle $u^{-}/|v^{-}|$ is greater or equal than $[(1+\kappa)\mu^{2}+\kappa-1]/(2\mu)$. For example, if $\mu = 0.5$ and $\kappa = 0.8$, the energy increase would occur if $u^{-}/|v^{-}| > 0.25$, *i.e.*, if the incidence angle is greater than about 14°. Of course, to remedy this outcome the model of material particle must be abandoned by including in the analysis the shape and rotational inertia of the small object.

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