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*Proving maximum principles using tent method**

Abstract

We present the tent theory that gives a powerful tool for proving a variety of maximum principles. New, shorter and simpler proofs of the main theorems of this theory are given. The efficiency of tent method is demonstrated in the proof of Mayer's optimal control problem with the class of piecewise continuous controls.

Key words: convex set, separability, cone, duality, optimality

1. Introduction

The tent method, developed by V.G. Boltyanski in his works published during 1972-1975, is a powerful tool in proving necessary conditions in the theory of extremal problems. In the first place this concerns theorems of the Pontryagin Maximum Principle type, which give necessary conditions for optimal control problems. In [4] and [5] Boltyanski gives a complete survey of his investigations related to the tent theory.

In [11] the survey of the tent theory is presented. New, shorter and simpler proofs of the main theorems are given. Using the tent method, in [12] and [13] maximum principles for the optimal control problems on bounded and on unbounded intervals with the class of measurable controls are proved.

In this article we present the tent theory with some modifications compared to [11]. We apply the tent method to prove maximum principle for the optimal control problem with the class of piecewise continuous controls.

The prerequisites for understanding this article are: basic courses in mathematical analysis and linear algebra, main facts about convex sets (which can be found in the first two chapters of [16]), Brouwer's fixed point theorem and the theory of ordinary differential equations [1, section 2.5].

In Section 1 we present some facts about convex sets in Euclidean spaces. Section 3 is concerned with duality of convex cones. These two sections are given here for the completeness sake. The results given there are known and can be found in [9] and [8].

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The base of tent theory is the theorem on separability of convex cones. In Section 2 we generalize the notion of separability of a finite family of convex cones to the notion of separability of a finite family of convex sets. In Section 4 we give a new proof of the theorem on separability of a finite family of convex cones, which is shorter and simpler than the old one (see [2] and [5]).

In Section 5 the notions of the tent and of the local tent are introduced in an equivalent, but slightly different form than those introduced by Boltyanski. A new proof of the main theorem in tent theory, shorter and simpler than those in [3] and [5], is given. In this proof we use only Brouwer's fixed point theorem, while in the old proofs a much more complicated topological tool was used.

In Section 6 we apply the tent method to prove maximum principle for the optimal control problem with the class of piecewise continuous controls. Mayer' s problem is considered. Since Bolza's problem can be reduced to Mayer's problem, the obtained result is of general nature.

In his more recent works (see [6] and [7]), Boltyanski modified the tent theory in order to obtain a method for solving extremal problems in the infinite dimensional case. However, our experience in applications of the "old" finite dimensional tent theory in proving different maximum principles is a very satisfactory one. This is the reason why we concentrate our attention on it.

2. Relative interior of a convex set

Let C be a convex set in Euclidean space X. The affine hull of a set C is the smallest flat, in the inclusion sense, in which C is contained. It is denoted by $\operatorname{aff} C$. The relative interior of a set C is defined by

$$\operatorname{ri} C = \operatorname{int}_{\operatorname{aff} C} C.$$

Theorem 2.1. *The relative interior of a convex set C in Euclidean space X is a convex nonempty set.*

Proof. The convexity of the set $\operatorname{ri} C$ follows from the fact that the interior of a convex set is convex too. Let $a_0 \in C$ and let $a_1 - a_0, a_2 - a_0, \dots, a_s - a_0$ be the maximal subset of linearly independent vectors of the set $C - a_0$. A simplex S with vertices at points $a_0, a_1, a_2, \dots, a_s$ is a subset of C and obviously it has a nonempty interior with respect to $\operatorname{aff} C$. It follows that $\operatorname{ri} C \neq \emptyset$.

The immediate consequence of the preceding theorem is

Theorem 2.2. *Let C be a convex set in Euclidean space X. Then*

$$\operatorname{ri} C = \operatorname{core}_{\operatorname{aff} C} C.$$

It follows that $a \in \operatorname{ri} C$ if and only if every line l, containing the point a and having at least one more common point with the set C, contains an open segment s, such that $a \in s \subseteq C$.

Theorem 2.3. *Let C_i, $i = 1, 2, \dots, m$, be convex sets in Euclidean space X. If*

$$\bigcap_{i=1}^{m} \operatorname{ri} C_i \neq \emptyset,$$

then

a) $\operatorname{aff} \bigcap_{i=1}^{m} C_i = \bigcap_{i=1}^{m} \operatorname{aff} C_i$,

b) $\operatorname{ri} \bigcap_{i=1}^{m} C_i = \bigcap_{i=1}^{m} \operatorname{ri} C_i$.

Proof. From $\bigcap_{i=1}^{m} C_i \subseteq \bigcap_{i=1}^{m} \operatorname{aff} C_i$ it follows that

$$\operatorname{aff} \bigcap_{i=1}^{m} C_i \subseteq \bigcap_{i=1}^{m} \operatorname{aff} C_i.$$

Suppose $x \in \bigcap_{i=1}^{m} \operatorname{ri} C_i$. Let l be a line satisfying the condition $x \in l \subseteq \bigcap_{i=1}^{m} \operatorname{aff} C_i$. There exist open segments $s_i \subseteq l$, $i = 1, 2, \dots m$, such that $x \in s_i \subseteq C_i$. Then $s = \bigcap_{i=1}^{m} s_i$ is an open segment on the line l satisfying $x \in s \subseteq \bigcap_{i=1}^{m} C_i$. It follows that $l \subseteq \operatorname{aff} \bigcap_{i=1}^{m} C_i$. Since $\bigcap_{i=1}^{m} \operatorname{aff} C_i$ is the union of lines laying in it and passing through x, then

$$\operatorname{aff} \bigcap_{i=1}^{m} C_i \supseteq \bigcap_{i=1}^{m} \operatorname{aff} C_i.$$

Thus the proof of a) is completed. From previous considerations it also follows that $x \in \operatorname{ri} \bigcap_{i=1}^{m} C_i$, and since x is an arbitrary point from $\bigcap_{i=1}^{m} \operatorname{ri} C_i$, then

$$\operatorname{ri} \bigcap_{i=1}^{m} C_i \supseteq \bigcap_{i=1}^{m} \operatorname{ri} C_i.$$

Let $a \in \bigcap_{i=1}^m \operatorname{ri} C_i$. For $x \in \operatorname{ri} \bigcap_{i=1}^m C_i$ there exists a point $u \in \bigcap_{i=1}^m C_i$ such that $x \in [a, u)$. Since $a \in \operatorname{ri} C_i$ and $u \in C_i$ then $x \in \operatorname{ri} C_i$, $i = 1, 2, \ldots, m$, and therefore $x \in \bigcap_{i=1}^m \operatorname{ri} C_i$. It follows that

$$\operatorname{ri} \bigcap_{i=1}^{m} C_i \subseteq \bigcap_{i=1}^{m} \operatorname{ri} C_i.$$

The proof of b) is completed.

Theorem 2.4. *Let X and Y be Euclidean spaces, let $A \in L(X;Y)$ and let C be a convex set in X. Then*

a) $\operatorname{aff} AC = A \operatorname{aff} C$,

b) $\operatorname{ri} AC = A \operatorname{ri} C$.

Proof. From $AC \subseteq A \operatorname{aff} C$ if follows that

$$\operatorname{aff} AC \subseteq A \operatorname{aff} C.$$

Suppose $x \in \operatorname{ri} C$, $y = Ax$. Let q be a line in Y satisfying the condition $y \in q \subseteq A \operatorname{aff} C$. Then there exists a line p in X such that $x \in p \subseteq \operatorname{aff} C$ and $Ap = q$. On the line p there exists an open segment s such that $x \in s \subseteq C$. The open segment As on the line q satisfies $y \in As \subseteq AC$. It follows that $q \subseteq \operatorname{aff} AC$. Since $A \operatorname{aff} C$ is the union of lines laying in it and passing through y, then

$$\operatorname{aff} AC \supseteq A \operatorname{aff} C.$$

Thus the proof of a) is completed. From previous considerations it also follows that $y \in \operatorname{ri} AC$, and since y is an arbitrary point from $A \operatorname{ri} C$, then

$$\operatorname{ri} AC \supseteq A \operatorname{ri} C.$$

Let $a \in \operatorname{ri} C$, $b = Aa \in AC$. Suppose $y \in \operatorname{ri} AC$. Then there exists a point $v \in AC$ such that $y \in [b, v)$. Let $u \in C$ be a point which satisfies the condition $Au = v$. Since $A[a, u) = [b, v)$, then there exists a point $x \in [a, u)$ for which $Ax = y$. Since $a \in \operatorname{ri} C$ and $u \in C$, then $x \in \operatorname{ri} C$, and therefore $y \in A \operatorname{ri} C$. It follows that

$$\operatorname{ri} AC \subseteq A \operatorname{ri} C.$$

The proof of b) is completed.

Theorem 2.5. *Let X and Y be Euclidean spaces, let $A \in L(X;Y)$ and let C be a convex set in Y. If the operator A is surjective, then*

a) $\operatorname{aff} A^{-1}C = A^{-1} \operatorname{aff} C$,

b) $\operatorname{ri} A^{-1}C = A^{-1} \operatorname{ri} C$.

Proof. From $A^{-1}C \subseteq A^{-1} \operatorname{aff} C$ it follows that

$$\operatorname{aff} A^{-1}C \subseteq A^{-1} \operatorname{aff} C.$$

Suppose $x \in A^{-1} \operatorname{ri} C$, $y = Ax$. Let p be a line in X satisfying the condition $x \in p \subseteq A^{-1} \operatorname{aff} C$. If $Ap = \{y\}$, then obviously $p \subseteq A^{-1}C$. Let $Ap = q$, where q is a line in Y. Since $y \in q \subseteq \operatorname{aff} C$ and $y \in riC$, then there exists an open segment s on the line p such that $x \in s \subseteq A^{-1}C$ (i.e. $y \in As \subseteq C$). It follows (in both cases) that $p \subseteq \operatorname{aff} A^{-1}C$. Since $A^{-1} \operatorname{aff} C$ is the union of lines laying in it and passing through x, then

$$\operatorname{aff} A^{-1}C \supseteq A^{-1} \operatorname{aff} C.$$

Thus the proof of a) is completed. From previous considerations it also follows that $x \in \operatorname{ri} A^{-1}C$, and since x is an arbitrary point from $A^{-1} \operatorname{ri} C$, then

$$\operatorname{ri} A^{-1}C \supseteq A^{-1} \operatorname{ri} C.$$

Let $a \in A^{-1} \operatorname{ri} C$. Suppose $x \in \operatorname{ri} A^{-1}C$. Then there exists a point $u \in A^{-1}C$ such that $x \in [a, u)$. Since $Ax \in [Aa, Au)$, $Aa \in \operatorname{ri} C$ and $Au \in C$, then $Ax \in \operatorname{ri} C$, i.e. $x \in A^{-1} \operatorname{ri} C$. It follows that

$$\operatorname{ri} A^{-1}C \subseteq A^{-1} \operatorname{ri} C.$$

The proof of b) is completed.

Theorem 2.6. *Let X_i be Euclidean spaces and let $C_i \subseteq X_i$ be convex sets, $i = 1, 2, \ldots, m$. Then*

a) $\operatorname{aff} \prod_{i=1}^{m} C_i = \prod_{i=1}^{m} \operatorname{aff} C_i$,

b) $\operatorname{ri} \prod_{i=1}^{m} C_i = \prod_{i=1}^{m} \operatorname{ri} C_i$.

Proof. According to Theorems 2.3 and 2.5 we have:

$$\operatorname{aff} \prod_{i=1}^{m} C_i = \operatorname{aff} \bigcap_{i=1}^{m} P_i^{-1} C_i = \bigcap_{i=1}^{m} \operatorname{aff} P_i^{-1} C_i = \bigcap_{i=1}^{m} P_i^{-1} \operatorname{aff} C_i = \prod_{i=1}^{m} \operatorname{aff} C_i.$$

$$\operatorname{ri} \prod_{i=1}^{m} C_i = \operatorname{ri} \bigcap_{i=1}^{m} P_i^{-1} C_i = \bigcap_{i=1}^{m} \operatorname{ri} P_i^{-1} C_i = \bigcap_{i=1}^{m} P_i^{-1} \operatorname{ri} C_i = \prod_{i=1}^{m} \operatorname{ri} C_i.$$

(P_i, $i = 1, 2, \ldots, m$, are projections of $X_1 \times X_2 \times \ldots \times X_m$ onto X_i.)

3. Separability of convex sets

Let C_i, $i = 1, 2, \ldots, m$, be convex sets in Euclidean space X. These convex sets are said to be separable if there exists a hyperplane which separates one of the them from the intersection of remaining ones.

A flat in a vector space is a set obtained by translation of some subspace. The translation is not unique, but the subspace is uniquely determined. This subspace is called the direction of the flat.

Theorem 3.1. *Convex sets C_i, $i = 1, 2, \ldots, m$, in Euclidean space X are not separable if and only if the following conditions are satisfied:*

a) $\displaystyle\bigcap_{i=1}^{m} \operatorname{ri} C_i \neq \emptyset$,

b) $\displaystyle Y_i + \bigcap_{j\neq i} Y_j = X$, $i = 1, 2, \ldots, m$, *where Y_i is the direction of the affine hull of the set C_i, $i = 1, 2, \ldots, m$.*

Proof. Let H be a hyperplane which separates sets C_m and $\bigcap_{i=1}^{m-1} C_i$ and let a) be fulfilled. Since

$$\operatorname{ri} C_m \cap \operatorname{ri} \bigcap_{i=1}^{m-1} C_i = \operatorname{ri} C_m \cap \bigcap_{i=1}^{m-1} \operatorname{ri} C_i = \bigcap_{i=1}^{m} \operatorname{ri} C_i \neq \emptyset,$$

then

$$C_m \subseteq H, \quad \bigcap_{i=1}^{m-1} C_i \subseteq H,$$

and therefore

$$\operatorname{aff} C_m \subseteq H, \quad \operatorname{aff} \bigcap_{i=1}^{m-1} C_i = \bigcap_{i=1}^{m-1} \operatorname{aff} C_i \subseteq H.$$

It follows that

$$Y_m + \bigcap_{j=1}^{m-1} Y_j \neq X.$$

Suppose that the condition a) is not satisfied. Let k be a natural number satisfying

$$\bigcap_{i=1}^{k-1} \operatorname{ri} C_i \neq \emptyset, \quad \bigcap_{i=1}^{k} \operatorname{ri} C_i = \emptyset.$$

Non empty convex sets $\operatorname{ri} C_k$ and $\operatorname{ri} \bigcap_{i=1}^{k-1} C_i = \bigcap_{i=1}^{k-1} \operatorname{ri} C_i$ are disjoint and therefore there exists a hyperplane H separating them. The hyperplane H separates C_k and $\bigcap_{i \neq k} C_i$.

Suppose that the condition a) is satisfied and that

$$Y_m + \bigcap_{j=1}^{m-1} Y_j \neq X.$$

There exists a hypersubspace Y containing Y_m and $\bigcap_{j=1}^{m-1} Y_j$. The hyperplane H, which is parallel to Y and which intersects $\bigcap_{j=1}^{m} C_j$, contains $\operatorname{aff} C_m$ and $\operatorname{aff} \bigcap_{i=1}^{m-1} C_i$. This hyperplane separates sets C_m and $\bigcap_{i=1}^{m-1} C_i$.

Lemma 3.1. *Let Y_i, $i = 1, 2, \ldots, m$, be subspaces of Euclidean space X. The following conditions are equivalent:*

a) $Y_i + \bigcap_{j \neq i} Y_j = X$, $i = 1, 2, \ldots, m$,

b) $\sum_{i=1}^{m} \bigcap_{j \neq i} Y_j = X$,

c) $\bigcap_{i} (x_i + Y_i)$, *for any* $x_i \in X$, $i = 1, 2, \ldots, m$,

d) $Y + Z = X^m$, *where* $Y = \prod_{i=1}^{m} Y_i$ *and* $Z = \{(z, z, \ldots, z) \in X^m \mid z \in X\}$.

Proof. We shall prove by induction on m that a) implies b). For $m = 2$, the conditions a) and b) are equal. Suppose that the statement is true for some integer $m - 1$, $m > 2$. Now we prove that the statement is true for m. Suppose that subspaces Y_i, $i = 1, 2, \ldots, m$, of Euclidean space X satisfy condition a). Subspaces $Z_i = Y_i \cap Y_m$, $i = 1, 2, \ldots, m - 1$, of Euclidean space Y_m satisfy condition a):

$$Z_i + \bigcap_{\substack{j=1 \\ j \neq i}}^{m-1} Z_j = Y_i \cap Y_m + \bigcap_{\substack{j=1 \\ j \neq i}}^{m-1} Y_j \cap Y_m = Y_i \cap Y_m + \bigcap_{\substack{j=1 \\ j \neq i}}^{m} Y_j =$$

$$= \left(Y_i + \bigcap_{\substack{j=1 \\ j \neq i}}^{m} Y_j \right) \cap Y_m = X \cap Y_m = Y_m.$$

By induction hypothesis we obtain

$$\sum_{i=1}^{m-1} \bigcap_{\substack{j=1 \\ j\neq i}}^{m-1} Z_j = Y_m,$$

i.e.

$$\sum_{i=1}^{m-1} \bigcap_{\substack{j=1 \\ j\neq i}}^{m} Y_j = Y_m.$$

Now we have

$$\sum_{i=1}^{m} \bigcap_{\substack{j=1 \\ j\neq i}}^{m} Y_j = \sum_{i=1}^{m-1} \bigcap_{\substack{j=1 \\ j\neq i}}^{m} Y_j + \bigcap_{j=1}^{m-1} Y_j = Y_m + \bigcap_{j=1}^{m-1} Y_j = X.$$

Suppose that b) is fulfilled. Than we have

$$Y_i + \bigcap_{j\neq i} Y_j \supseteq \sum_{\substack{k=1 \\ k\neq i}}^{m} \bigcap_{j\neq k} Y_j + \bigcap_{j\neq i} Y_j = \sum_{k=1}^{m} \bigcap_{j\neq k} Y_j = X,$$

wherefrom it follows that a) holds.

Suppose that a) holds. Let $x_i \in X$, $i = 1, 2, \ldots, m$. Each vector x_i, $i = 1, 2, \ldots, m$, can be represented as $x_i = y_i + z_i$, where $y_i \in Y_i$ and $z_i \in \bigcap_{j\neq i} Y_j$. Put $z = z_1 + z_2 + \ldots + z_m$. Then $z - x_i = \sum_{j\neq i} z_j - y_i \in Y_i$, and therefore $z \in x_i + Y_i$, $i = 1, 2, \ldots, m$. It follows that c) is fulfilled.

Suppose that c) holds. Let $x \in X$. From c) we obtain

$$(x + Y_i) \cap \bigcap_{j\neq i} Y_j \neq \emptyset, \quad i = 1, 2, \ldots, m.$$

It follows that there exist vectors $y_i \in Y_i$ and $z_i \in \bigcap_{j\neq i} Y_j$ such that $x + y_i = z_i$. Since $x = -y_i + z_i$, we have that $x \in Y_i + \bigcap_{j\neq i} Y_j$. So, a) is fulfilled.

Suppose that c) holds. Let $(x_1, x_2, \ldots, x_m) \in X^m$. Since the set $\bigcap_i (x_i + Y_i)$ is nonempty, there exists $z \in X$ such that $z \in x_i + Y_i$, $i = 1, 2, \ldots, m$. It follows that x_i can be represented as $x_i = y_i + z$, where $y_i \in Y_i$, for each $i = 1, 2, \ldots, m$. Therefore $(x_1, x_2, \ldots, x_m) = (y_1, y_2, \ldots, y_m) + (z, z, \ldots, z) \in Y + Z$. It follows that $Y + Z = X^m$, i.e. that d) holds.

Suppose that d) holds. Let $x_i \in X$, $i = 1, 2, \ldots, m$. Since $(x_1, x_2, \ldots, x_m) \in Y + Z$, there exist vectors $y_i \in Y_i$, $i = 1, 2, \ldots, m$, and $z \in X$, such that

$x_i = y_i + z$, $i = 1, 2, \ldots, m$. It follows that the vector z belongs to each of the sets $x_i + Y_i$, $i = 1, 2, \ldots, m$. Therefore the set $\bigcap_i (x_i + Y_i)$ is nonempty, i.e. c) is fulfilled.

Theorem 3.2. *Convex sets C_i, $i = 1, 2, \ldots, m$, in Euclidean space X are not separable if and only if sets $C = \prod_{i=1}^m C_i$ and $Z = \{(z, z, \ldots, z) \in X^m \mid z \in X\}$ are not separable.*

Proof. The condition a) of Theorem 3.1 for the sets C_i, $i = 1, 2, \ldots, m$, is equivalent to

$$\operatorname{ri} C \cap \operatorname{ri} Z = \prod_{i=1}^m \operatorname{ri} C_i \cap Z \neq \emptyset,$$

and this is in fact the condition a) of Theorem 3.1 for the sets C and Z. According to Theorem 2.6 we conclude that Y is the direction of the set C. Having this fact in mind and using the preceding lemma we obtain that the condition b) of Theorem 3.1 for the sets C_i, $i = 1, 2, \ldots, m$, is equivalent to the condition b) of Theorem 3.1 for the sets C and Z. From previous facts and from Theorem 3.1 it follows that sets C_i, $i = 1, 2, \ldots, m$, are separable if and only if C and Z are separable.

Theorem 3.3. *Convex sets C_i, $i = 1, 2, \ldots, m$, in Euclidean space X are separable if sets $\bigcap_{i=1}^k C_i$ and $\bigcap_{i=k+1}^m C_i$ are separable.*

Proof. Suppose that convex sets C_i, $i = 1, 2, \ldots, m$, are not separable. Then the condition a) from Theorem 3.1 and the condition b) from Lemma 3.1 are satisfied. Since

$$\bigcap_{i=1}^k \operatorname{ri} C_i \neq \emptyset, \qquad \bigcap_{i=k+1}^m \operatorname{ri} C_i \neq \emptyset,$$

we have

$$\operatorname{ri} \bigcap_{i=1}^k C_i = \bigcap_{i=1}^k \operatorname{ri} C_i, \quad \operatorname{ri} \bigcap_{i=k+1}^m C_i = \bigcap_{i=k+1}^m \operatorname{ri} C_i,$$

and so

$$\operatorname{ri} \bigcap_{i=1}^k C_i \cap \operatorname{ri} \bigcap_{i=k+1}^m C_i = \bigcap_{i=1}^m \operatorname{ri} C_i \neq \emptyset.$$

Also we have

$$\operatorname{aff} \bigcap_{i=1}^k C_i = \bigcap_{i=1}^k \operatorname{aff} C_i, \quad \operatorname{aff} \bigcap_{i=k+1}^m C_i = \bigcap_{i=k+1}^m \operatorname{aff} C_i.$$

It follows that $\bigcap_{i=1}^k Y_i$ and $\bigcap_{i=k+1}^m Y_i$ are directions of affine hulls of sets $\bigcap_{i=1}^k C_i$ and $\bigcap_{i=k+1}^m C_i$. Since

$$\bigcap_{i=1}^{k} Y_i \supseteq \sum_{i=k+1}^{m} \bigcap_{\substack{j=1\\ j\neq i}}^{m} Y_j, \qquad \bigcap_{i=k+1}^{m} Y_i \supseteq \sum_{i=1}^{k} \bigcap_{\substack{j=1\\ j\neq i}}^{m} Y_j,$$

and

$$\sum_{i=k+1}^{m} \bigcap_{\substack{j=1\\ j\neq i}}^{m} Y_j + \sum_{i=1}^{k} \bigcap_{\substack{j=1\\ j\neq i}}^{m} Y_j = \sum_{i=1}^{m} \bigcap_{\substack{j=1\\ j\neq i}}^{m} Y_j = X,$$

we have

$$\bigcap_{i=1}^{k} Y_i + \bigcap_{i=k+1}^{m} Y_i = X.$$

According to Theorem 3.1, the sets $\bigcap_{i=1}^k C_i$ and $\bigcap_{i=k+1}^m C_i$ are not separable.

Examples: Let $x_i^* = X^* \setminus \{0\}$, $i = 1, 2, \ldots, m$, be linear functionals.

1. *Halfspaces*

$$K_i = \{x \in X \mid x_i^* x \leq 0\}, \quad i = 1, 2, \ldots, m,$$

are not separable if and only if

$$\bigcap_{i=1}^{m} \operatorname{int} K_i = \bigcap_{i=1}^{m} \{x \in X \mid x_i^* x < 0\} \neq \emptyset.$$

2. *Hyperplanes*

$$K_i = \{x \in X \mid x_i^* x = 0\} = \ker x_i^*, \quad i = 1, 2, \ldots, m,$$

are not separable if and only if linear functionals x_i^*, $i = 1, 2, \ldots, m$, are linearly independent.

3. *Cones*

$$K_i = \{x \in X \mid x_i^* x \leq 0\}, \quad i = 1, 2, \ldots, k,$$
$$K_i = \{x \in X \mid x_i^* x = 0\}, \quad i = k+1, \ldots, m,$$

are not separable if and only if linear functionals x_i^*, $i = k+1, \ldots, m$, are linearly independent and

$$\bigcap_{i=1}^{k} \operatorname{int} K_i \cap \bigcap_{i=k+1}^{m} K_i = \bigcap_{i=1}^{k} \{x \in X \mid x_i^* x < 0\} \cap \bigcap_{i=k+1}^{m} \{x \in X \mid x_i^* x = 0\} \neq \emptyset.$$

4. Dual cones

We shall only consider cones with vertex at zero. Let K be a cone in Euclidean space X. Its dual cone is defined in the following way:

$$K^* = \{x^* \in X^* \mid (\forall x \in K) x^* x \leq 0\}.$$

Theorem 4.1. *Let K, $K_1, K_2, \ldots, K_m$ be cones in Euclidean space X. Then:*

a) *K^* is a closed convex cone,*

b) *$K^{**} = \operatorname{cconv} K$,*

c) *$K_1 \subseteq K_2 \Leftrightarrow K_1^* \supseteq K_2^*$,*

d) $\left(\bigcup_{i=1}^{m} K_i\right)^* = \bigcap_{i=1}^{m} K_i^*$,

e) *if cones K_i, $i = 1, 2, \ldots, m$, are convex and closed, then*

$$\left(\bigcap_{i=1}^{m} K_i\right)^* = \operatorname{cl} \sum_{i=1}^{m} K_i^*.$$

Proof. a) K^* can be represented as the intersection of closed halfspaces:

$$K^* = \bigcap_{x \in K} \{x^* \in X^* \mid x^* x \leq 0\}.$$

b) Let $x \in K$. Every $x^* \in K^*$ satisfies $x^* x \leq 0$. Therefore $x \in K^{**}$. It follows that $K \subseteq K^{**}$, and this together with a) gives

$$\operatorname{cconv} K \subseteq K^{**}.$$

Let $x \notin \operatorname{cconv} K$. The point x and the set $\operatorname{cconv} K$ can be strictly separated. Therefore there exists $x^* \in K^*$ such that $x^* x > 0$. It follows that $x \notin K^{**}$. Hence

$$K^{**} \subseteq \operatorname{cconv} K.$$

c) Suppose $K_1 \subseteq K_2$. Then

$$K_1^* = \bigcap_{x \in K_1} \{x^* \in X^* \mid x^* x \leq 0\} \supseteq \bigcap_{x \in K_2} \{x^* \in X^* \mid x^* x \leq 0\} = K_2^*.$$

d)

$$\left(\bigcup_{i=1}^{m} K_i\right)^* = \bigcap_{x\in\bigcup K_i} \{x^* \in X^* \mid x^*x \le 0\} = \bigcap_{i=1}^{m} \bigcap_{x\in K_i} \{x^* \in X^* \mid x^*x \le 0\} = \bigcap_{i=1}^{m} K_i^*.$$

e) Let cones K_i, $i = 1, 2, \ldots, m$, be convex and closed. Then

$$K_i = \operatorname{cconv} K_i = K_i^{**}, i = 1, 2, \ldots, m.$$

Therefore

$$\left(\bigcap_{i=1}^{m} K_i\right)^* = \left(\bigcap_{i=1}^{m} K_i^{**}\right)^* = \left(\bigcup_{i=1}^{m} K_i^*\right)^{**} =$$

$$= \operatorname{cconv} \bigcup_{i=1}^{m} K_i^* = \operatorname{cl} \sum_{i=1}^{m} K_i^*.$$

Examples:

1. *If*

$$K = \{x \in X \mid x^*x \le 0\},$$

where $x^* \in X^*$, $x^* \neq 0$, *then*

$$K^* = \{\lambda x^* \mid \lambda \ge 0\}.$$

2. *If*

$$K = \{x \in X \mid x^*x = 0\} = \ker x^*,$$

where $x^* \in X^*$, $x^* \neq 0$, *then*

$$K^* = \{\lambda x^* \mid \lambda \in R\}.$$

5. Separability of convex cones

Theorem 5.1. *Convex cones K_i, $i = 1, 2, \ldots, m$, in Euclidean space X are separable if and only if there exist linear functionals $x_i^* \in K_i^*$, $i = 1, 2, \ldots, m$, such that at least one of them is different from zero and that $\sum_{i=1}^{m} x_i^* = 0$.*

Proof. According to Theorem 3.2, cones K_i, $i = 1, 2, \ldots, m$, are separable if and only if the cone K and the subspace Z of X^m are separable. The cone

K and the subspace Z are separable if and only if there exists a functional $x^* \in X^{m*}$ such that $x^* \neq 0$, $x^* \in K^*$ and $x^* \in Z^\perp$. If $x^* \in X^{m*}$, then there exist functionals $x_i^* \in X^*$, $i = 1, 2, \ldots, m$, such that $x^*x = \sum_{i=1}^m x_i^* x_i$, for $x = (x_1, x_2, \ldots, x_m) \in X^m$. It is easy to see that

1. $x^* \neq 0$ if and only if $x_i^* \neq 0$ for at least one of $i = 1, 2, \ldots m$;
2. $x^* \in K^*$ if and only if $x_i^* \in K_i^*$ for $i = 1, 2, \ldots m$;
3. $x^* \in Z^\perp$ if and only if $\sum_{i=1}^m x_i^* = 0$.

The proof is completed.

6. Tents and local tents

Let X be Euclidean space, let $M \subseteq X$, let $a \in M$ and let K be a convex cone in X. The cone K is said to be a tent of the set M at the point a if there exists a continuous function $\varphi : K \to M$ such that $\varphi(0) = a$ and $\varphi'(0) = I$.

Remark. Usually, in the definition of the strong derivative of a function $f : D \to Y$ at the point $a \in D$, it is assumed that a is the interior point of the set D. We shall not require this assumption in this section. Theorems related to the derivative which we shall use (a theorem on the derivative of a sum of functions, a theorem on the derivative of a composition of functions...) hold without the assumption that a function is defined in the neighborhood of a point in which we consider the derivative.

Let X be Euclidean space, let $M \subseteq X$, let $a \in M$ and let K be a convex cone in X. Suppose that there exist a neighborhood of zero U and a continuous function $\psi : K \cap U \to M$ such that $\psi(0) = a$ and $\psi'(0) = I$. We can assume that $U = B[0, r]$. Let $\pi : K \to K \cap U$ be defined by

$$\pi(x) = \begin{cases} x & , \quad x \in K \cap U \\ \dfrac{rx}{\|x\|} & , \quad x \in K \setminus U \end{cases} .$$

The function $\varphi = \psi \circ \pi : K \to M$ satisfies conditions $\varphi(0) = a$ and $\varphi'(0) = I$. So K is a tent of the set M at the point a.

Let X be Euclidean space, let $M \subseteq X$, let $a \in M$ and let K be a convex cone in X. The cone K is said to be a local tent of the set M at the point a if for each point $x \in \operatorname{ri} K$ there exists a tent $K_x \subseteq K$ of the set M at the point a, such that $x \in \operatorname{ri} K_x$ and $\operatorname{aff} K_x = \operatorname{aff} K$.

Theorem 6.1. *Let X and Y be Euclidean spaces, let $M \subseteq X$ and let K be a local tent of the set M at the point a. Further, let $f : M \to Y$ be a continuous* *function, differentiable at the point a. Then $f'(a)K$ is the local tent of the set $f(M)$ at the point $f(a)$.*

Proof. Suppose
$$y \in \operatorname{ri} f'(a)K = f'(a) \operatorname{ri} K.$$
There exists a point $x \in \operatorname{ri} K$ such that $f'(a)x = y$. Let $K_x \subseteq K$ be a tent of the set M at the point a, which satisfies conditions $x \in \operatorname{ri} K_x$ and $\operatorname{aff} K_x = \operatorname{aff} K$; suppose $\varphi : K_x \to M$ is continuous, $\varphi(0) = a$ and $\varphi'(0) = I$. Since
$$y \in f'(a) \operatorname{ri} K_x = \operatorname{ri} f'(a)K_x,$$
then there exist linearly independent vectors $y_i \in f'(a)K_x$, $i = 1, 2, \ldots, m$, generating the convex cone K_y, such that $y \in \operatorname{ri} K_y$ and $\operatorname{aff} K_y = \operatorname{aff} f'(a)K_x = \operatorname{aff} f'(a)K$. Suppose vectors $x_i \in K_x$, $i = 1, 2, \ldots, m$, satisfy $f'(a)x_i = y_i$, $i = 1, 2, \ldots, m$. Let $A : K_y \to K_x$ be the linear operator determined by $Ay_i = x_i$, $i = 1, 2, \ldots, m$. The function $\psi = f \circ \varphi \circ A$ maps K_y into $f(M)$, it is continuous, differentiable at zero and satisfies $\psi(0) = f(a)$ and $\psi'(0) = I$.

Theorem 6.2. *Let M_i, $i = 1, 2, \ldots, m$, be subsets of Euclidean space X, let $\bigcap_{i=1}^{m} M_i = \{a\}$ and let K_i, $i = 1, 2, \ldots, m$, be their local tents at the point a. If at least one of the cones K_i, $i = 1, 2, \ldots, m$, is not a flat, then they are separable.*

Proof. Without loss of generality we shall suppose that $a = 0$.

Suppose cones K_i, $i = 1, 2, \ldots, m$, are not separable. According to Theorem 3.1 and to the definition of the local tent we conclude that it is enough to consider the case when K_i are tents of the sets M_i at the point 0, $i = 1, 2, \ldots, m$.

Put
$$M = M_1 \times M_2 \times \ldots \times M_m,$$
$$K = K_1 \times K_2 \times \ldots \times K_m,$$
$$Z = \{(x, x, \ldots, x) \in X^m \mid x \in X\}.$$
K is a convex cone in X^m and K is not a flat. Z is a subspace of the Euclidean space X^m, $M \cap Z = \{0\}$. Let $\varphi_i : K_i \to M_i$ be continuous functions satisfying $\varphi_i(0) = 0$ and $\varphi_i'(0) = I$. The function $\varphi : K \to M$ defined by
$$\varphi(x_1, x_2, \ldots, x_m) = (\varphi_1(x_1), \varphi_2(x_2), \ldots, \varphi_m(x_m))$$
is continuous and satisfies conditions $\varphi(0) = 0$ and $\varphi'(0) = I$. It follows that K is a tent of the set M at the point 0.

According to Theorem 3.2 the cone K and the subspace Z are not separable. According to Theorem 3.1 there exist a point $z_0 \in Z \cap \operatorname{ri} K$ and a subspace $Y \subseteq \operatorname{aff} K$, such that $Y \dot{+} Z = X^m$. Obviously $z_0 \notin Y$.

Let $P : X^m \to Y$ be the projection operator parallel to the subspace Z. Let $B = B[0, r]$ be a closed ball in Y satisfying $z_0 + B \subseteq K$. Further, let ϵ, δ and λ be positive numbers satisfying:

$$\epsilon \leq \frac{r}{\|P\|(r + \|z_0\|)}, \quad \epsilon < 1,$$

$$x \in K, \|x\| \leq \delta \Rightarrow \|\varphi(x) - x\| \leq \epsilon \|x\|,$$

$$\lambda = \frac{\delta}{r + \|z_0\|}.$$

The function $f : B \to Y$ defined by

$$f(y) = y - \frac{1}{\lambda} P\varphi(\lambda(y + z_0))$$

is continuous. From

$$\|f(y)\| = \left\| \frac{1}{\lambda} P\varphi(\lambda(y + z_0)) - \frac{1}{\lambda} P(\lambda(y + z_0)) \right\|$$

$$\leq \frac{1}{\lambda} \|P\| \|\varphi(\lambda(y + z_0)) - \lambda(y + z_0)\|$$

$$\leq \frac{1}{\lambda} \|P\| \epsilon \lambda \|y + z_0\| \leq \|P\| \epsilon (r + \|z_0\|) \leq r$$

it follows that $f(B) \subseteq B$. According to the Brouwer's fixed point theorem, there exists $y_0 \in B$ such that $f(y_0) = y_0$, i.e. $\varphi(\lambda(y_0 + z_0)) \in Z$. Let $x_0 = \varphi(\lambda(y_0 + z_0))$. From

$$\|x_0 - \lambda(y_0 + z_0)\| \leq \epsilon \|\lambda(y_0 + z_0)\| < \|\lambda(y_0 + z_0)\|$$

it follows that $x_0 \neq 0$. Since $x_0 \in M \cap Z = \{0\}$, then $x_0 = 0$. Contradiction!

Examples:

1. *Let $f : D \to R$ be a differentiable function at the point a. If $f(a) = 0$ and $f'(a) \neq 0$, then*

$$K = \{x \in X \mid f'(a)x \leq 0\}$$

is a local tent of the set

$$M = \{x \in D \mid f(x) < 0\} \cup \{a\}$$

at the point a.

Proof. Without loss of generality we can suppose that $a = 0$. Let $z \in \operatorname{int} K$, i.e. let $f'(0)z < 0$. Suppose B is a closed ball with the center z laying in the $\operatorname{int} K$. The function $f'(0)x/\|x\|$ is continuous on B and takes on it negative values only. Therefore there exists an $\epsilon > 0$ such that $f'(0)x \le -\epsilon\|x\|$ for every $x \in B$. This inequality holds also on the convex cone K_z generated by the ball B. Let $U \subseteq D$ be a neighborhood of zero such that

$$|f(x) - f'(0)x| < \epsilon\|x\|, \quad x \in U \setminus \{0\}.$$

Then we have

$$f(x) < f'(0)x + \epsilon\|x\| \le -\epsilon\|x\| + \epsilon\|x\| = 0.$$

for $x \in K_z \cap U$, $x \neq 0$. It follows that $K_z \cap U \subseteq M$.

2. *Let $f : D \to R$ be a strictly differentiable function at the point a. If $f(a) = 0$ and $f'(a) \neq 0$, then*

$$K = \{x \in X \mid f'(a)x = 0\} = \ker f'(a)$$

is a tent of the set

$$M = \{x \in D \mid f(x) = 0\}$$

at the point a.

Proof. The function $g(x, y) = f(a+x+y)$ is defined in the neighborhood of the point $(0, 0)$ and satisfies all assumptions from the implicit function theorem (see §2.3 in [1]). Therefore there exist a neighborhood U of 0, a continuous function $\gamma : U \to X$ and a number C satisfying

$$f(a + x + \gamma(x)) = 0,$$

$$\|\gamma(x)\| \le C\|f(a + x)\|,$$

for every $x \in U$. The function $\varphi : K \cap U \to M$, defined by $\varphi(x) = a+x+\gamma(x)$, is continuous and satisfies conditions $\varphi(0) = a$ and $\varphi'(0) = I$.

7. Mayer's problem of optimal control

Let U be a topological space, let G be an open set in $R \times R^n$ and let W be an open set in $R \times R^n \times R \times R^n$. Moreover, let $f(t, x, u) : X \times U \to R^n$ and

$l(t_0, x_0, t_1, x_1) : W \to R^{m+1}$ be continuous functions. The coordinate functions of l are denoted by l_i, $i = 0, 1, \ldots, m$.

The set of processes P is defined in the following way

$$P = \{(x(\cdot), u(\cdot), t_0, t_1) \in \bar{C}^1([t_0, t_1], R^n) \times \bar{C}([t_0, t_1], U) \times R \times R \mid$$

$$(t_0, x(t_0), t_1, x(t_1)) \in W\}.$$

We denote by $\bar{C}^1([t_0, t_1], R^n)$ and $\bar{C}([t_0, t_1], U)$ the class of piecewise smooth functions mapping the interval $[t_0, t_1]$ into the phase space R^n and the class of piecewise continuous functions mapping the interval $[t_0, t_1]$ into the control set U, respectively.

Mayer's problem of optimal control is the following extremal problem on the set of processes P:

$$\begin{aligned} l_0(t_0, x(t_0), t_1, x(t_1)) \to inf; \quad & l_i(t_0, x(t_0), t_1, x(t_1)) \leq 0 \text{ for } i = 1, \ldots, k, \\ & l_i(t_0, x(t_0), t_1, x(t_1)) = 0 \text{ for } i = k+1, \ldots, m, \\ & \dot{x}(t) = f(t, x(t), u(t)) \text{ for all } t \in T, \end{aligned}$$

where T is the set of points from the interval $[t_0, t_1]$, in which the control function $u(\cdot)$ is continuous.

The process $(\hat{x}(\cdot), \hat{u}(\cdot), \hat{t}_0, \hat{t}_1)$ is optimal in the strong sense if it is admissible and if there exists an $\epsilon > 0$ such that

$$l_0(t_0, x(t_0), t_1, x(t_1)) \geq l_0(\hat{t}_0, \hat{x}(\hat{t}_0), \hat{t}_1, \hat{x}(\hat{t}_1))$$

for each admissible process $(x(\cdot), u(\cdot), t_0, t_1)$, satisfying

$$|t_0 - \hat{t}_0|, |t_1 - \hat{t}_1| < \epsilon,$$

$$\|x(t) - \hat{x}(t)\| < \epsilon \text{ for every } t \in [\hat{t}_0, \hat{t}_1] \cap [t_0, t_1].$$

Theorem 7.1. *Suppose $f(t, x, u)$ has continuous derivative according to the variable x and $l(t_0, x_0, t_1, x_1)$ is continuously differentiable. If the process $(\hat{x}(\cdot), \hat{u}(\cdot), \hat{t}_0, \hat{t}_1)$ is optimal in the strong sense, then there exist $\hat{\lambda} = (\hat{\lambda}_0, \hat{\lambda}_1, \ldots, \hat{\lambda}_m) \in R^{m+1*}$, and a piecewise smooth function $\hat{p}(\cdot) : [\hat{t}_0, \hat{t}_1] \to R^{n*}$, satisfying the following conditions:*

0. $\hat{\lambda} \neq 0$,
1. $\hat{\lambda}_i \geq 0$ *for* $i = 0, 1, \ldots, k$,

2. $\hat{\lambda}_i l_i(\hat{t}_0, \hat{x}(\hat{t}_0), \hat{t}_1, \hat{x}(\hat{t}_1)) = 0$ *for* $i = 1, \ldots, k$,
3. $\dot{\hat{p}}(t) = -\hat{p}(t) f_x(t, \hat{x}(t), \hat{u}(t))$ *for all* $t \in \hat{T}$,
4. $\hat{p}(\hat{t}_0) = \hat{\lambda} l_{x_0}(\hat{t}_0, \hat{x}(\hat{t}_0), \hat{t}_1, \hat{x}(\hat{t}_1))$,
5. $\hat{p}(\hat{t}_1) = -\hat{\lambda} l_{x_1}(\hat{t}_0, \hat{x}(\hat{t}_0), \hat{t}_1, \hat{x}(\hat{t}_1))$,
6. $\hat{p}(\hat{t}_0) f(\hat{t}_0, \hat{x}(\hat{t}_0), \hat{u}(\hat{t}_0)) = -\hat{\lambda} l_{t_0}(\hat{t}_0, \hat{x}(\hat{t}_0), \hat{t}_1, \hat{x}(\hat{t}_1))$,
7. $\hat{p}(\hat{t}_1) f(\hat{t}_1, \hat{x}(\hat{t}_1), \hat{u}(\hat{t}_1)) = \hat{\lambda} l_{t_1}(\hat{t}_0, \hat{x}(\hat{t}_0), \hat{t}_1, \hat{x}(\hat{t}_1))$,
8. $\hat{p}(t) f(t, \hat{x}(t), \hat{u}(t)) \le \hat{p}(t) f(t, \hat{x}(t), u)$ *for all* $t \in \hat{T}$, $u \in U$,

where $\hat{T}$ *is the set of points from the interval* $[\hat{t}_0, \hat{t}_1]$ *in which the control function* $\hat{u}(\cdot)$ *is continuous.*

Proof. Put $\hat{x}_0 = \hat{x}(\hat{t}_0)$, $\hat{x}_1 = \hat{x}(\hat{t}_1)$. Without loss of generality we can assume that $l_0(\hat{t}_0, \hat{x}_0, \hat{t}_1, \hat{x}_1) = 0$. If $l_0(\hat{t}_0, \hat{x}_0, \hat{t}_1, \hat{x}_1) \neq 0$, we can put $l_0(t_0, x_0, t_1, x_1) - l_0(\hat{t}_0, \hat{x}_0, \hat{t}_1, \hat{x}_1)$ instead of the function $l_0(t_0, x_0, t_1, x_1)$. We can also assume that all constraints are active, i.e. that $l_i(\hat{t}_0, \hat{x}_0, \hat{t}_1, \hat{x}_1) = 0$, $i = 1, \ldots, k$. If $l_j(\hat{t}_0, \hat{x}_0, \hat{t}_1, \hat{x}_1) < 0$ for some j, $1 \le j \le k$, then we can replace the set W by an open subset containing the point $(\hat{t}_0, \hat{x}_0, \hat{t}_1, \hat{x}_1)$ on which the function $l_j(t_0, x_0, t_1, x_1)$ is negative, and we can put $\hat{\lambda}_j = 0$ and drop the corresponding constraint from further considerations. Condition 2 will be satisfied.

If for some j, $0 \le j \le m$, we have $l'_j(\hat{t}_0, \hat{x}_0, \hat{t}_1, \hat{x}_1) = 0$, the assertion of our theorem holds. It is enough to take $\hat{\lambda}_j = 1$, $\hat{\lambda}_i = 0$ for $i \neq j$, $i = 0, 1, \ldots, m$, and $\hat{p}(\cdot) = 0$. Further on, we assume that $l'_i(\hat{t}_0, \hat{x}_0, \hat{t}_1, \hat{x}_1) \neq 0$, $i = 0, 1, \ldots, m$.

Convex cones

$K_i = \{(t_0, x_0, t_1, x_1) \in R \times R^n \times R \times R^n \mid l'_i(\hat{t}_0, \hat{x}_0, \hat{t}_1, \hat{x}_1)(t_0, x_0, t_1, x_1) \le 0\}$, for $i = 0, 1, \ldots, k$,

$K_i = \{(t_0, x_0, t_1, x_1) \in R \times R^n \times R \times R^n \mid l'_i(\hat{t}_0, \hat{x}_0, \hat{t}_1, \hat{x}_1)(t_0, x_0, t_1, x_1) = 0\}$, for $i = k+1, \ldots, m$

are local tents of the sets

$M_0 = \{(t_0, x_0, t_1, x_1) \in W \mid l_0(t_0, x_0, t_1, x_1) < 0\} \cup \{(\hat{t}_0, \hat{x}_0, \hat{t}_1, \hat{x}_1)\}$,

$M_i = \{(t_0, x_0, t_1, x_1) \in W \mid l_i(t_0, x_0, t_1, x_1) \le 0\}$, $i = 1, \ldots, k$,

$M_i = \{(t_0, x_0, t_1, x_1) \in W \mid l_i(t_0, x_0, t_1, x_1) = 0\}$, $i = k+1, \ldots, m$,

at the point $(\hat{t}_0, \hat{x}_0, \hat{t}_1, \hat{x}_1)$ (see Section 6).

Let $M \subseteq W$ be the set of ordered quadruples $(t_0, x(t_0), t_1, x(t_1))$, where $(x(\cdot), u(\cdot), t_0, t_1)$ are processes satisfying the differential equation

$$\dot{x}(t) = f(t, x(t), u(t)),$$

such that

$$|t_0 - \hat{t}_0|, |t_1 - \hat{t}_1| < \epsilon,$$

$$\|x(t) - \hat{x}(t)\| < \epsilon \text{ for every } t \in [\hat{t}_0, \hat{t}_1] \cap [t_0, t_1].$$

Lemma 7.1. *Let $R(t,\tau)$ be the resolvent of the differential equation*

$$\dot{x}(t) = f_x(t, \hat{x}(t), \hat{u}(t))x(t).$$

The convex cone K in $R \times R^n \times R \times R^n$, generated by vectors

$\pm(0, 0, 1, f(\hat{t}_1, \hat{x}(\hat{t}_1), \hat{u}(\hat{t}_1)))$,

$\pm(1, 0, 0, -R(\hat{t}_1, \hat{t}_0)f(\hat{t}_0, \hat{x}(\hat{t}_0), \hat{u}(\hat{t}_0)))$,

$(0, x_0, 0, R(\hat{t}_1, \hat{t}_0)x_0)$, $x_0 \in R^n$,

$(0, 0, 0, R(\hat{t}_1, \tau)(f(\tau, \hat{x}(\tau), u) - f(\tau, \hat{x}(\tau), \hat{u}(\tau)))$, *for all* $\tau \in \hat{T}$, $u \in U$,

is a local tent of the set M at the point $(\hat{t}_0, \hat{x}_0, \hat{t}_1, \hat{x}_1)$.

This lemma will be proved later.

The intersection of the sets M, M_i, $i = 0, 1, \ldots, m$, contains one point only: $(\hat{t}_0, \hat{x}_0, \hat{t}_1, \hat{x}_1)$. Convex cones K, K_i, $i = 0, 1, \ldots, m$, are their local tents at that point. The cone K_0 is not a flat. According to Theorem 6.2, convex cones K, K_i, $i = 0, 1, \ldots, m$, are separable. According to Theorem 5.1, there exist linear functionals $\hat{x}_i^* \in K_i^*$, $i = 0, 1, \ldots, m$, such that at least one of them is different from zero and $-\sum_{i=0}^m \hat{x}_i^* \in K^*$. We have $\hat{x}_i^* = \hat{\lambda}_i l_i'(\hat{t}_0, \hat{x}_0, \hat{t}_1, \hat{x}_1)$, $i = 0, 1, \ldots, m$, where $\hat{\lambda}_i \geq 0$ for $i = 0, 1, \ldots, k$ (see examples in section 4). The vector $\hat{\lambda} = (\hat{\lambda}_0, \hat{\lambda}_1, \ldots, \hat{\lambda}_m) \in R^{m+1*}$ satisfies the conditions 0 and 1. Since $-\hat{\lambda}l'(\hat{t}_0, \hat{x}_0, \hat{t}_1, \hat{x}_1) \in K^*$, then

(1) $\hat{\lambda}l_{t_1}(\hat{t}_0, \hat{x}_0, \hat{t}_1, \hat{x}_1) + \hat{\lambda}l_{x_1}(\hat{t}_0, \hat{x}_0, \hat{t}_1, \hat{x}_1)f(\hat{t}_1, \hat{x}(\hat{t}_1), \hat{u}(\hat{t}_1)) = 0$,

(2) $\hat{\lambda}l_{t_0}(\hat{t}_0, \hat{x}_0, \hat{t}_1, \hat{x}_1) - \hat{\lambda}l_{x_1}(\hat{t}_0, \hat{x}_0, \hat{t}_1, \hat{x}_1)R(\hat{t}_1, \hat{t}_0)f(\hat{t}_0, \hat{x}(\hat{t}_0), \hat{u}(\hat{t}_0)) = 0$,

(3) $\hat{\lambda}l_{x_0}(\hat{t}_0, \hat{x}_0, \hat{t}_1, \hat{x}_1) + \hat{\lambda}l_{x_1}(\hat{t}_0, \hat{x}_0, \hat{t}_1, \hat{x}_1)R(\hat{t}_1, \hat{t}_0) = 0$,

(4) $\hat{\lambda}l_{x_1}(\hat{t}_0, \hat{x}_0, \hat{t}_1, \hat{x}_1)R(\hat{t}_1, \tau)(f(\tau, \hat{x}(\tau), u) - \hat{f}(\tau, \hat{x}(\tau), \hat{u}(\tau)) \geq 0$, for all $\tau \in \hat{T}$, $u \in U$.

Define the piecewise smooth function $\hat{p}(\cdot) : [\hat{t}_0, \hat{t}_1] \to R^{n*}$ by

$$\hat{p}(t) = -\hat{\lambda}l_{x_1}(\hat{x}_0, \hat{x}_1)R(\hat{t}_1, t).$$

Obviously it satisfies the conditions 3 and 5. Conditions 4, 6, 7 i 8 are equivalent to (3), (2), (1) i (4).

Proof of Lemma 7.1. It is enough to prove that, for any finite family of pairs $(\tau_j, u_j) \in \hat{T} \times U$, $j = 1, \ldots, q$, the convex cone in $R \times R^n \times R \times R^n$ generated by vectors $(0, x_0, 0, R(\hat{t}_1, \hat{t}_0)x_0)$, $x_0 \in R^n$,

$\pm(1, 0, 0, -R(\hat{t}_1, \hat{t}_0)f(\hat{t}_0, \hat{x}(\hat{t}_0), \hat{u}(\hat{t}_0)))$,
$\pm(0, 0, 1, f(\hat{t}_1, \hat{x}(\hat{t}_1), \hat{u}(\hat{t}_1)))$,
$(0, 0, 0, R(\hat{t}_1, \tau_j)(f(\tau_j, \hat{x}(\tau_j), u_j) - f(\tau_j, \hat{x}(\tau_j), \hat{u}_j(\tau_j))))$, $j = 1, \ldots, q$,

is the local tent of the set M at the point $(\hat{t}_0, \hat{x}_0, \hat{t}_1, \hat{x}_1)$.

Extend the control $\hat{u}(\cdot)$ onto the whole set of real numbers by $\hat{u}(t) = \hat{u}(t_0)$ for $t < t_0$ and $\hat{u}(t) = \hat{u}(t_1)$ for $t > t_1$. There exist a closed interval I, containing $[\hat{t}_0, \hat{t}_1]$ in its interior, and an extension of $\hat{x}(\cdot)$ defined on I satisfying

$$\dot{x}(t) = f(t, x(t), \bar{u}(t)) \text{ for all } t \in \hat{I}.$$

We can suppose that $\hat{t}_0 < \tau_1 \leq \tau_2 \leq \cdots \leq \tau_q < \hat{t}_1$. Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_q)$ be a q-tuple of non negative real numbers, $\sigma = \sum \alpha_j$. If $\alpha_1, \alpha_2, \ldots, \alpha_q$ are sufficiently small, then the following intervals

$$I_j = [\tau_j + j\sigma, \tau_j + j\sigma + \alpha_j), \quad j = 1, 2, \ldots, q,$$

are disjoint, and all of them lie in I. The control $u(\cdot) : I \to U$, defined by

$$u(t) = \begin{cases} \hat{u}(t) & , \quad t \in I \setminus \cup I_j \\ u_j & , \quad t \in I_j \end{cases}$$

we shall call the needle variation of the control $\bar{u}(\cdot)$.

It can be proved that:

a) If $|t_0 - \hat{t}_0|$, $\|x_0 - \hat{x}_0\|$, $\alpha_1, \alpha_2, \ldots, \alpha_q$ are sufficiently small, then the problem

$$\dot{x}(t) = f(t, x(t), u(t)), \quad x(t_0) = x_0$$

has a solution on the interval I, denoted by $x(t, t_0, x_0, \alpha)$.

b) If $t_0 \to \hat{t}_0$, $x_0 \to \hat{x}_0$, $\alpha_1, \alpha_2, \ldots, \alpha_q \to 0_+$, then $x(t, t_0, x_0, \alpha)$ uniformly converges to $\bar{x}(t)$ on I.

Let $x(t, \tau, \xi)$ be the maximal solution of the differential equation

$$\dot{x}(t) = f(x(t), \bar{u}(t))$$

and let $x_j(t, \tau, \xi)$ be maximal solutions of differential equations

$$\dot{x}(t) = f(x(t), u_j), \quad j = 1, 2, \ldots, q.$$

If $|t_1 - \hat{t}_1|$, $|t_0 - \hat{t}_0|$, $\|x_0 - \hat{x}_0\|$, $\alpha_1, \alpha_2, \ldots, \alpha_q$ are sufficiently small, then the function $x(t_1, t_0, x_0, \alpha)$ can be represented as the composition of the following sequence of functions:

$$x(\tau_1 + 1\sigma, t_0, x_0),$$

$$x_1(\tau_1 + 1\sigma + \alpha_1, \tau_1 + 1\sigma, \cdot),$$

$$x(\tau_2 + 2\sigma, \tau_1 + 1\sigma + \alpha_1, \cdot),$$

$$\ldots$$

$$x_q(\tau_q + q\sigma + \alpha_q, \tau_q + q\sigma, \cdot),$$

$$x(t_1, \tau_q + q\sigma + \alpha_q, \cdot).$$

We can conclude that the function $x(t_1, t_0, x_0, \alpha)$ can be continuously extended in some open neighborhood of the point $(\hat{t}_1, \hat{t}_0, \hat{x}_0, 0)$, so that the extension is differentiable at $(\hat{t}_1, \hat{t}_0, \hat{x}_0, 0)$. Partial derivatives of this extension are given by:

$$\frac{\partial}{\partial t_1} x(\hat{t}_1, \hat{t}_0, \hat{x}_0, 0) = f(\hat{t}_0, \hat{x}(\hat{t}_0), \hat{u}(\hat{t}_0))$$

$$\frac{\partial}{\partial t_0} x(\hat{t}_1, \hat{t}_0, \hat{x}_0, 0) = -R(\hat{t}_1, \hat{t}_0) f(\hat{t}_0, \hat{x}(\hat{t}_0), \hat{u}(\hat{t}_0))$$

$$\frac{\partial}{\partial x_0} x(\hat{t}_1, \hat{t}_0, \hat{x}_0, 0) = R(\hat{t}_1, \hat{t}_0),$$

$$\frac{\partial}{\partial \alpha_j} x(\hat{t}_1, \hat{t}_0, \hat{x}_0, 0) = R(\hat{t}_1, \tau_j)[f(\tau_j, \hat{x}(\tau_j), u_j) - f(\tau_j, \hat{x}(\tau_j), \hat{u}(\tau_j))],$$

where $R(t, \tau)$ is the resolvent of the equation

$$\dot{x}(t) = f_x(t, \hat{x}(t), \hat{u}(t)) x(t).$$

First three formulas arise from $x(t_1, t_0, x_0, \alpha)|_{\alpha=0} = x(t_1, t_0, x_0)$, and the last one we can derive from

$$x(\hat{t}_1, \hat{t}_0, \hat{x}_0, 0, \ldots, 0, \alpha_j, 0, \ldots, 0) =$$

$$= x(\hat{t}_1, \tau_j + (j+1)\alpha_j, x_j(\tau_j + (j+1)\alpha_j, \tau_j + j\alpha_j, x(\tau_j + j\alpha_j, \hat{t}_0, \hat{x}_0))),$$

in the following way:

$$\frac{\partial}{\partial \alpha_j} x(\hat{t}_1, \hat{t}_0, \hat{x}_0, 0) = -R(\hat{t}_1, \tau_j) f(\tau_j, \hat{x}(\tau_j), \hat{u}(\tau_j))(j+1) +$$

$$+R(\hat{t}_1,\tau_j)[f(\tau_j,\hat{x}(\tau_j),u_j)(j+1)-f(\tau_j,\hat{x}(\tau_j),u_j)\cdot j+f(\tau_j,\hat{x}(\tau_j),\hat{u}(\tau_j))\cdot j]=$$
$$=R(\hat{t}_1,\tau_j)[f(\tau_j,\hat{x}(\tau_j),u_j)-f(\tau_j,\hat{x}(\tau_j),\hat{u}(\tau_j))].$$

There exists $\delta>0$ such that $(t_1,t_0,x_0,\alpha)\to(t_0,x_0,t_1,x(t_1,t_0,x_0,\alpha))$ maps $R\times R\times R^n\times[0,+\infty)^q\cap B((\hat{t}_1,\hat{t}_0,\hat{x}_0,0),\delta)$ into M. The same function maps the point $(\hat{t}_1,\hat{t}_0,\hat{x}_0,0)$ to the point $(\hat{t}_0,\hat{x}_0,\hat{t}_1,\hat{x}_1)$. Obviously $R\times R\times R^n\times[0,+\infty)^q$ is a tent of the set $R\times R\times R^n\times[0,+\infty)^q\cap B((\hat{t}_1,\hat{t}_0,\hat{x}_0,0),\delta)$ at the point $(\hat{t}_0,\hat{x}_0,\hat{t}_1,\hat{x}_1)$. The cone $R\times R\times R^n\times[0,+\infty)^q$ is generated by vectors

$$(\pm1,0,0,0),$$

$$(0,\pm1,0,0),$$

$$(0,0,x_0,0),\quad x_0\in R^n,$$

$$(0,0,0,e_j),\quad j=1,2,\ldots,q.$$

The derivative of the considered function at the point $(\hat{t}_1,\hat{t}_0,\hat{x}_0,0)$ maps the cone $R\times R\times R^n\times[0,+\infty)^q$ into the cone described in the formulation this lemma. According to Theorem 6.1, this cone is a local tent of the set M at the point $(\hat{t}_0,\hat{x}_0,\hat{t}_1,\hat{x}_1)$.

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